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COMPACT PERTURBATIONS OF CONTROLLED SYSTEMS

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Abstract. In this article we study the controllability properties of general compactly perturbed exactly controlled linear systems with admissible control operators. Firstly, we show that approximate and exact controllability are equivalent properties for such systems. Then, and more importantly, we provide for the perturbed system a complete characterization of the set of reachable states in terms of the Fattorini-Hautus test. The results rely on the Peetre lemma.

1. **Introduction and main results.** In this work we study the exact controllability property of general compactly perturbed controlled linear systems using a compactness-uniqueness approach. This technique was notably used in the pioneering work [19] to establish the exponential decay of the solutions to some hyperbolic equations. On the other hand, the first controllability results using this method were obtained in [23, 15] for a plate equation and in [24] for a perturbed wave equation. Wether one wants to establish a stability result or a controllability result, one is led in both cases to prove estimates, energy estimates or observability inequalities. For a perturbed system, a general procedure is to start by the known estimate satisfied by the unperturbed system and to try to derive the desired estimate, up to some "lower order terms" that we would like to remove. The compactness-uniqueness argument then reduces the task of absorbing these additional terms to a unique continuation property for the perturbed system. We should point out that, despite the numerous applications of this flexible method to successfully establish the controllability of systems governed by partial differential equations (see e.g. [23, 15, 24, 2, 12, 4, 8, 14], etc.), no systematic treatment has been provided so far for compactly perturbed systems, by which we mean that there is no abstract result available in the literature that covers all type of systems, regardless the nature of the equations we are considering (plate, wave, etc.). This will be the first point of the present paper to fill this gap (Theorem 1.1 below). Then, and more importantly, we

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considerably improve this result by establishing an explicit characterization of the set of reachable states for the perturbed system (Theorem 1.2 below). This characterization is given in terms of the Fattorini-Hautus test, which is a far weaker kind of unique continuation property than the approximate controllability. Our result shows in particular that this test is actually sufficient to ensure the exact controllability of the perturbed system (Corollary 1.3 below). We illustrate this corollary and the significance of the Fattorini-Hautus test with an application to the exact controllability of a Schrödinger equation perturbed by a non local term (Theorem 3.1 below). Finally, we conclude this work by mentioning that the techniques used in this paper are not only limited to the study of the controllability of compactly perturbed controlled systems (Theorem 4.1 below). The proofs of the main results of this article are based on the Peetre Lemma, introduced in [18], which is in fact the root of compactness-uniqueness methods.

Let us now introduce some notations and recall some basic facts about the controllability of abstract linear evolution equations. We refer to the excellent textbook [22] for the proofs of the statements below. Let H and U be two (real or complex) Hilbert spaces, let $A : D(A) \subset H \longrightarrow H$ be the generator of a C_0-semigroup $(S_A(t))_{t\geq 0}$ on H and let $B \in \mathcal{L}(U, D(A^*)')$. For $T \geq 0$ let $\Phi_T \in \mathcal{L}(L^2(0,+\infty;U), D(A^*)')$ be the input map of (A,B), that is the linear operator defined for every $u \in L^2(0,+\infty;U)$ by

$$\Phi_T u = \int_0^T S_A(T-s)Bu(s)\,ds.$$

We assume that B is admissible for A, which means that $\operatorname{Im}\Phi_T \subset H$ for some (and hence all) $T > 0$. From this assumption it follows that $\Phi_T \in \mathcal{L}(L^2(0,+\infty;U), H)$. Its adjoint $\Phi_T^* \in \mathcal{L}(H, L^2(0,+\infty;U))$ is the unique continuous linear extension to H of the map $z \in D(A^*) \mapsto B^*S_A(T-\cdot)^*z \in L^2(0,+\infty;U)$, where $B^*S_A(T-\cdot)^*z$ is extended by zero outside $(0,T)$ (in particular, $\Phi_T^*z(t) = 0$ for a.e. $t > T$). Let us now consider the abstract evolution system

$$\begin{cases} \dfrac{d}{dt}y &= Ay + Bu, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1}$$

where $T > 0$ is the time of control, $y^0 \in H$ is the initial data, y is the state and $u \in L^2(0,T;U)$ is the control. Since B is admissible for A, system (1) is well-posed: for every $y^0 \in H$ and every $u \in L^2(0,T;U)$, there exists a unique solution $y \in C^0([0,T];H)$ to system (1) given by the Duhamel formula

$$y(t) = S_A(t)y^0 + \Phi_t u, \quad \forall t \geq 0.$$

The regularity of the solution allows us to consider control problems for system (1). We say that system (1) or (A,B) is:

- exactly controllable in time T if, for every $y^0, y^1 \in H$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution y to system (1) satisfies $y(T) = y^1$.
- approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in H$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution y to system (1) satisfies $\left\|y(T) - y^1\right\|_H \leq \varepsilon$.

Clearly, exact controllability in time T implies approximate controllability in the same time. The set $\operatorname{Im}\Phi_T$ (resp. $\overline{\operatorname{Im}\Phi_T}$) is called the set of exactly (resp. approximately) reachable states in time T. Therefore, (A,B) is exactly (resp. approximately) controllable in time T if, and only if, $\operatorname{Im}\Phi_T = H$ (resp. $\overline{\operatorname{Im}\Phi_T} = H$). It is also well-known that the controllability has a dual concept named observability. More precisely, (A,B) is exactly controllable in time T if, and only if, there exists $C>0$ such that

$$\|z\|_H^2 \leq C\int_0^T \|\Phi_T^* z(t)\|_U^2\, dt, \quad \forall z \in H, \tag{2}$$

and (A,B) is approximately controllable in time T if, and only if,

$$\Big(\Phi_T^* z(t) = 0, \quad \text{a.e. } t\in(0,T)\Big) \Longrightarrow z=0, \quad \forall z\in H. \tag{3}$$

Let us now state the main results of this paper. The first one simply unifies previous results available in the literature under a general semigroup setting:

Theorem 1.1. *Let H and U be two (real or complex) Hilbert spaces. Let $A_0 : D(A_0)\subset H \longrightarrow H$ be the generator of a C_0-semigroup on H and let us consider $B\in\mathcal{L}(U, D(A_0^*)')$ an admissible control operator for A_0. Let $K\in\mathcal{L}(H)$ and let us form the unbounded operator $A_K = A_0+K$ with $D(A_K)=D(A_0)$ [1]. We assume that:*

(i) There exists $T_0>0$ such that (A_0,B) is exactly controllable in time T_0.
(ii) K is compact.
(iii) (A_K,B) is approximately controllable in time T_0.

Then, (A_K,B) is exactly controllable in time T_0.

The second and most important result of the present paper shows that we can even give a very precise characterization of the reachable states for the perturbed system, if we allow the time of control to be slightly longer:

Theorem 1.2. *Let H and U be complex Hilbert spaces and let A_0, A_K and B be defined as in Theorem 1.1. For $T\geq 0$ let $\Phi_T\in\mathcal{L}(L^2(0,+\infty;U),H)$ be the input map of (A_K,B). Let σ_F be the set given by*

$$\sigma_F = \left\{\lambda\in\mathbb{C}, \quad \ker(\lambda - A_K^*)\cap\ker B^* \neq \{0\}\right\},$$

and for every $\lambda\in\mathbb{C}$ let E_λ be the subspace of H defined by

$$E_\lambda = \left\{ z \in \bigcup_{m=1}^{+\infty} \ker(\lambda - A_K^*)^m, \quad B^*(\lambda - A_K^*)^m z = 0, \quad \forall m\in\mathbb{N}\right\}.$$

Then, under the assumptions (i) and (ii) of Theorem 1.1, the set σ_F is finite, E_λ is finite dimensional for every $\lambda\in\mathbb{C}$, and we have

$$\operatorname{Im}\Phi_T = \left(\bigoplus_{\lambda\in\sigma_F} E_\lambda\right)^{\perp}, \quad \forall T>T_0.$$

This second result shows in particular that the approximate controllability assumption (iii) of Theorem 1.1 can be weakened to the Fattorini-Hautus test:

[1] A_K is then the generator of a C_0-semigroup on H and B is also admissible for A_K, see below.

Corollary 1.3. *Let H and U be complex Hilbert spaces and let A_0, A_K and B be defined as in Theorem 1.1. Then, under the assumptions (i) and (ii) of Theorem 1.1, and if (A_K, B) satisfies the Fattorini-Hautus test:*

$$\ker(\lambda - A_K^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}, \tag{4}$$

then (A_K, B) is exactly controllable in time T for every $T > T_0$.

Remark 1.4. The assumption (ii) can be relaxed in some cases, see Remark 2.4 below. Besides, the abstract method we will develop in this paper is not only limited to the study of the controllability of perturbed controlled systems and it can be used to obtain the exact same conclusions as in the previous theorems for some (unperturbed) systems themselves. We explained it in Section 4 below.

Remark 1.5. It follows from Corollary 1.3 that, if (A_0, B) and (A_K, B) are two systems satisfying the Fattorini-Hautus test, and K is compact, then

$$\begin{aligned} &\inf \{T > 0,\ (A_0, B) \text{ is exactly controllable in time } T\} \\ &\qquad = \inf \{T > 0,\ (A_K, B) \text{ is exactly controllable in time } T\}. \end{aligned}$$

In other words, both systems share the same minimal time of control.

Corollary 1.3 shows that, in order to prove the exact controllability of a compactly perturbed system which is known to be exactly controllable, it is (necessary and) sufficient to only check the Fattorini-Hautus test (4). This result has been established in a particular case in [4, Theorem 5] for a perturbed plate equation with distributed controls. The Fattorini-Hautus test appears for the very first time in [7, Corollary 3.3] and it is also sometimes misleadingly known as the Hautus test in finite dimension, despite it has been introduced earlier by Fattorini, moreover in a much larger setting. In a complete abstract control theory framework, it is the sharpest sufficient condition one can hope for since it is always a necessary condition for the exact, null or approximate controllability, to hold in some time (this easily follows from the dual characterizations (2) or (3)). It is also nowadays well-known that this condition characterizes the approximate controllability of a large class of systems generated by analytic semigroups (see [7, 1]). In practice, the Fattorini-Hautus test can be checked by various techniques, such as Carleman estimates for stationary systems (see e.g. [4, 1]), or spectral analysis when this later technique is not available (see e.g. [3, 5] or the example of Section 3 below).

Let us mention that wether the Fattorini-Hautus test (4) remains sufficient or not to obtain the exact controllability of the perturbed system in time T_0 may depend on the systems under consideration. Therefore, both Theorem 1.1 and Corollary 1.3 are important. Obviously, Corollary 1.3 is a stronger result if we do not look for the best time. However, in some situations it might be important to preserve the same time of control (see e.g. [5, Theorem 1.1]).

Finally, let us point out that throughout this work we do not request any spectral properties whatsoever on the operators A_0 or A_K, contrary to the papers [11, 16] where the existence of a Riesz basis of generalized eigenvectors or related spectral properties are required.

The rest of this paper is organized as follows. In Section 2, we prove the main results of this work. In Section 3, we illustrate the abstract results with an application to the controllability of a perturbed Schrödinger equation. In Section 4, we point out that the techniques used in this paper are not only restricted to the study of the controllability of compactly perturbed controlled systems. Finally, we have included in Appendix A a proof of an estimate that is needed in the proof of our main results (especially for unbounded admissible control operators).

2. **Proofs of the results.** The proofs of Theorem 1.1 and Theorem 1.2 both rely on the Peetre Lemma (see [18, Lemma 3]):

Lemma 2.1. *Let B_1, B_2 be two Banach spaces and let $L \in \mathcal{L}(B_1, B_2)$. The following two conditions are equivalent:*

(a) There exist a Banach space B_3, a compact operator $P \in \mathcal{L}(B_1, B_3)$ and $\alpha > 0$ such that

$$\alpha \|z\|_{B_1} \leq \|Lz\|_{B_2} + \|Pz\|_{B_3}, \quad \forall z \in B_1. \tag{5}$$

(b) $\ker L$ is finite dimensional and $\operatorname{Im} L$ is closed.

Remark 2.2. It is well-known that $\ker L = \{0\}$ and $\operatorname{Im} L$ is closed if, and only if, there exists $\beta > 0$ such that (see e.g. [18, Lemma 4])

$$\beta \|z\|_{B_1} \leq \|Lz\|_{B_2}, \quad \forall z \in B_1.$$

Therefore, the compact term in (5) can be cancelled if one shows that $\ker L = \{0\}$.

Let us denote by $(S_{A_0}(t))_{t\geq 0}$ (resp. $(S_{A_K}(t))_{t\geq 0}$) the C_0-semigroup generated by A_0 (resp. A_K). For $T \geq 0$ let $\Psi_T \in \mathcal{L}(L^2(0,+\infty;U),H)$ (resp. $\Phi_T \in \mathcal{L}(L^2(0,+\infty;U),H)$) be the input map of (A_0,B) (resp. (A_K,B)). Assume now that (A_0,B) is exactly controllable in time T_0. Then, for every $T \geq T_0$, there exists $C > 0$ such that, for every $z \in H$,

$$\|z\|_H^2 \leq C \int_0^T \|\Psi_T^* z(t)\|_U^2 \, dt,$$

so that

$$\|z\|_H^2 \leq 2C \left(\int_0^T \|\Phi_T^* z(t)\|_U^2 \, dt + \int_0^T \|\Psi_T^* z(t) - \Phi_T^* z(t)\|_U^2 \, dt \right). \tag{6}$$

To prove that (A_K,B) is exactly controllable in time T, we would like to get rid of the last term in the right-hand side of (6). Therefore, we would like to apply Lemma 2.1 (see also Remark 2.2) to the operators $L = \Phi_T^*$ and $P = \Psi_T^* - \Phi_T^*$. Note that both operators are bounded linear operators since B is admissible for both A_0 and A_K (see below). To apply Lemma 2.1, we have to check that $\Psi_T^* - \Phi_T^*$ is compact.

Lemma 2.3. *The operator $\Psi_T^* - \Phi_T^* \in \mathcal{L}(H, L^2(0,+\infty;U))$ is compact for every $T > 0$.*

For the proof of Lemma 2.3 we need to recall the following estimate (see Appendix A): for every $T > 0$, there exists $C > 0$ such that, for every $f \in C^1([0,T];H)$,

$$\int_0^T \left\| B^* \int_0^t S_{A_0}(t-s)^* f(s)\, ds \right\|_U^2 dt \leq C \|f\|_{L^2(0,T;H)}^2. \tag{7}$$

This estimate holds because B is admissible for A_0 (for bounded operators $B \in \mathcal{L}(U,H)$ it is a straightforward consequence of the Cauchy-Schwarz inequality). Using the dual characterization of admissibility (see (27) below) and combining (7) with the identity (8) below we also see that, if B is admissible for A_0, then B is admissible for A_K as well.

Proof of Lemma 2.3. Let us first compute $\Psi_T^* - \Phi_T^*$. To this end, we recall the integral equation satisfied by semigroups of boundedly perturbed operators (see e.g. [6, Corollary III.1.7]), valid for every $z \in H$ and $t \in [0, T]$:

$$S_{A_K}(t)^* z = S_{A_0}(t)^* z + \int_0^t S_{A_0}(t-s)^* F z(s)\, ds, \tag{8}$$

where we introduced $F \in \mathcal{L}(H, L^2(0,T;H))$ defined for every $z \in H$ and every $s \in [0, T]$ by

$$Fz(s) = K^* S_{A_K}(s)^* z.$$

Note that $Fz \in C^1([0,T]; H)$ for $z \in D(A_K^*)$. Thus, we have $\int_0^t S_{A_0}(t-s)^* F z(s)\, ds \in D(A_0^*)$ for every $t \in (0,T)$ if $z \in D(A_K^*)$. This shows that each term in (8) actually belongs to $D(A_0^*)$ if $z \in D(A_0^*) = D(A_K^*)$. Therefore, we can apply B^* to obtain the following expression for $\Psi_T^* - \Phi_T^*$:

$$(\Psi_T^* - \Phi_T^*) z(t) = -B^* \int_0^{T-t} S_{A_0}(T-t-s)^* F z(s)\, ds,$$

for every $z \in D(A_0^*)$ and a.e. $t \in (0,T)$. Using now (7) there exists $C > 0$ such that

$$\|(\Psi_T^* - \Phi_T^*) z\|_{L^2(0,T;U)} \leq C \, \|Fz\|_{L^2(0,T;H)},$$

for every $z \in D(A_K^*)$, and thus for every $z \in H$ by density. To conclude the proof it only remains to show that F is compact. Since H is a Hilbert space, we will prove that, if $(z_n)_n \subset H$ is such that $z_n \to 0$ weakly in H as $n \to +\infty$, then $Fz_n \to 0$ strongly in $L^2(0,T;H)$ as $n \to +\infty$. Since $z_n \to 0$ weakly in H as $n \to +\infty$, using the strong (and therefore weak) continuity of semigroups on H, we obtain

$$S_{A_K}(s)^* z_n \xrightarrow[n \to +\infty]{} 0 \quad \text{weakly in } H, \quad \forall s \in [0,T].$$

Since K^* is compact, we obtain

$$K^* S_{A_K}(s)^* z_n \xrightarrow[n \to +\infty]{} 0 \quad \text{strongly in } H, \quad \forall s \in [0,T].$$

On the other hand, by the classical semigroup estimate, $(K^* S_{A_K}(s)^* z_n)_n$ is clearly uniformly bounded in H with respect to s and n. Therefore, the Lebesgue's dominated convergence theorem applies, so that $Fz_n \to 0$ strongly in $L^2(0,T;H)$ as $n \to +\infty$. This shows that F is compact. □

Remark 2.4. In all this work, the assumption (ii) that K is compact is only used to establish Lemma 2.3. Therefore, the results obtained in Theorem 1.1 and 1.2 remain valid if, instead of assuming that K is compact, we assume that the difference of the input maps is compact. This assumption is for instance satisfied by some first order hyperbolic systems (see e.g. [17, Thereom A]). This assumption is also satisfied if A_0 generates an immediately compact semigroup, as it is easily seen from the previous proof by changing the roles of A_0 and A_K and switching the convergence arguments at the end. This concerns for instance the Korteweg-de Vries equation studied in [20, Section 3]. Let us point out that this is only interesting if we consider unbounded control operators since the exact controllability is impossible if the semigroup is immediately compact and the control operator is bounded (see [21, Theorem 1.2]).

The proof of Theorem 1.1 is now a direct consequence of Lemma 2.1 and 2.3.

Proof of Theorem 1.1. The assumptions of Lemma 2.1 are satisfied for $L = \Phi_{T_0}^*$ and $P = \Psi_{T_0}^* - \Phi_{T_0}^*$. Therefore, $\operatorname{Im}\Phi_{T_0}$ is closed (we recall that $\operatorname{Im}\Phi_{T_0}$ is closed if, and only if, so is $\operatorname{Im}\Phi_{T_0}^*$) and it follows from the very definitions of the notions of controllability that (A_K, B) is then exactly controllable in time T_0 if, and only if, (A_K, B) is approximately controllable in time T_0. □

Remark 2.5. We see that we have actually obtained a more precise conclusion than the one stated in Theorem 1.1, namely that the targets that can be reached approximately can also be reached exactly (wether the whole system is controllable or not).

Note that so far we have used only the second part of the conclusion of Lemma 2.1. For the proof of Theorem 1.2 we need the following general result:

Lemma 2.6. *Let H and U be two complex Hilbert spaces. Let $A : D(A) \subset H \longrightarrow H$ be the generator of a C_0-semigroup on H and let $B \in \mathcal{L}(U, D(A^*)')$ be admissible for A. For $T \geq 0$ let $\Phi_T \in \mathcal{L}(L^2(0,+\infty;U), H)$ be the input map of (A, B). Let σ_F be the set given by*

$$\sigma_F = \{\lambda \in \mathbb{C}, \quad \ker(\lambda - A^*) \cap \ker B^* \neq \{0\}\}, \tag{9}$$

and for every $\lambda \in \mathbb{C}$ let E_λ be the subspace of H defined by

$$E_\lambda = \left\{ z \in \bigcup_{m=1}^{+\infty} \ker(\lambda - A^*)^m, \quad B^*(\lambda - A^*)^m z = 0, \quad \forall m \in \mathbb{N} \right\}. \tag{10}$$

Assume that there exists $T_0 > 0$ such that

$$\dim \ker \Phi_{T_0}^* < +\infty. \tag{11}$$

Then, the set σ_F is finite, E_λ is finite dimensional for every $\lambda \in \mathbb{C}$, and we have

$$\ker \Phi_T^* = \bigoplus_{\lambda \in \sigma_F} E_\lambda, \quad \forall T > T_0. \tag{12}$$

This result already appeared in the literature for some particular problems and in the particular case $\sigma_F = \emptyset$ (see e.g. [2, Theorem 3.8], [20, Lemma 3.4], [4, Theorem 5], etc.). To the best of our knowledge, the original ideas seem to trace back to [19, Proposition 2]. However, we will follow the more recent presentation done in the proof of [4, Theorem 5].

Remark 2.7. In the finite dimensional case $H = \mathbb{C}^n$ and $U = \mathbb{C}^m$ $(n, m \in \mathbb{N}^*)$ we recover the well-known fact that $\operatorname{Im}\Phi_T = \operatorname{Im}(B|AB|\cdots|A^{n-1}B)$ for every $T > 0$.

Proof of Lemma 2.6. Let us first prove that, for every $T > 0$ and $\lambda \in \mathbb{C}$, we have

$$\ker \Phi_T^* \supset E_\lambda. \tag{13}$$

Let then $z \in E_\lambda$. Thus, $z \in D((A^*)^\infty)$ and there exists $m \in \mathbb{N}^*$ such that

$$(\lambda - A^*)^m z = 0, \tag{14}$$

and

$$B^*(\lambda - A^*)^r z = 0, \quad \forall r \in \{0, \ldots, m-1\}. \tag{15}$$

Thanks to (14) we have, for every $t \geq 0$,

$$S_A(t)^* z = e^{\lambda t} \sum_{r=0}^{m-1} \frac{t^r}{r!} (A^* - \lambda)^r z.$$

Applying B^* and using (15) we obtain that $z \in \ker \Phi_T^*$. This establishes (13). Since the sum $\sum_{\lambda \in \sigma_F} E_\lambda$ is a direct sum, (13) implies that

$$\ker \Phi_T^* \supset \bigoplus_{\lambda \in \sigma_F} E_\lambda, \quad \forall T > 0. \tag{16}$$

In particular, by (11) we obtain that σ_F is finite and that E_λ is finite dimensional for every $\lambda \in \sigma_F$.

Let us now prove the reverse inclusion for $T > T_0$. The key point is to establish that

$$\ker \Phi_T^* \subset D(A^*). \tag{17}$$

Let us first recall that, for every $T_2 \geq T_1$, we have $\ker \Phi_{T_2}^* \subset \ker \Phi_{T_1}^*$ since

$$\int_0^{T_1} \left\|\Phi_{T_1}^* z(t)\right\|_U^2 \, dt \leq \int_0^{T_2} \left\|\Phi_{T_2}^* z(t)\right\|_U^2 \, dt, \quad \forall z \in H,$$

(this is easily proved for $z \in D(A^*)$ and thus remains true for $z \in H$ by density and continuity of Φ_T^* for any $T > 0$). Therefore, thanks to the assumption (11) we know that

$$\dim \ker \Phi_T^* < +\infty, \quad \forall T \geq T_0. \tag{18}$$

From now on, T is fixed such that $T > T_0$. Let $z \in \ker \Phi_T^*$. To prove that $z \in D(A^*)$, we have to show that, for any sequence $t_n > 0$ with $t_n \to 0$ as $n \to +\infty$, the sequence

$$u_n = \frac{S_A(t_n)^* z - z}{t_n}$$

converges in H as $n \to +\infty$. Let $\varepsilon \in (0, T - T_0]$ be fixed so that $T - \varepsilon \geq T_0$ and let $N \in \mathbb{N}$ be large enough so that $t_n < \varepsilon$ for every $n \geq N$. Let us first show that

$$u_n \in \ker \Phi_{T-\varepsilon}^*, \quad \forall n \geq N. \tag{19}$$

To this end, observe that, for $n \geq N$, we have (arguing by density as before)

$$\int_0^{T-\varepsilon} \left\|\Phi_{T-\varepsilon}^* S_A(t_n)^* z(t)\right\|_U^2 \, dt = \int_{\varepsilon - t_n}^{T - t_n} \left\|\Phi_T^* z(t)\right\|_U^2 \, dt.$$

This shows that $S_A(t_n)^* z \in \ker \Phi_{T-\varepsilon}^*$ for $n \geq N$ since $z \in \ker \Phi_T^*$. Thus, we have (19). Let now $\mu \in \rho(A^*)$ be fixed and let us introduce the following norm on $\ker \Phi_{T-\varepsilon}^*$:

$$\|z\|_{-1} = \left\|(\mu - A^*)^{-1} z\right\|_H.$$

Since $(\mu - A^*)^{-1} z \in D(A^*)$, we have

$$(\mu - A^*)^{-1} u_n = \frac{S_A(t_n)^* - \mathrm{Id}}{t_n} (\mu - A^*)^{-1} z \xrightarrow[n \to +\infty]{} A^*(\mu - A^*)^{-1} z \quad \text{ in } H.$$

Therefore, $(u_n)_{n \geq N}$ is a Cauchy sequence in $\ker \Phi_{T-\varepsilon}^*$ for the norm $\|\cdot\|_{-1}$. Since $\ker \Phi_{T-\varepsilon}^*$ is finite dimensional (by (18)), all the norms are equivalent on $\ker \Phi_{T-\varepsilon}^*$. Thus, $(u_n)_{n \geq N}$ is then a Cauchy sequence for the usual norm $\|\cdot\|_H$ as well and, as a result, converges for this norm. This shows that $z \in D(A^*)$ and establishes (17).

Thanks to (17) it is now easy to see that $\ker \Phi_T^*$ is stable by A^* and that we have

$$\ker \Phi_T^* \subset \ker B^*. \tag{20}$$

Indeed, let $z \in \ker \Phi_T^*$, that is

$$\Phi_T^* z(t) = 0, \quad \text{a.e. } t \in (0, T). \tag{21}$$

On the one hand, since $z \in D(A^*)$, we can differentiate this identity to obtain (see e.g. [22, Proposition 4.3.4])

$$\Phi_T^* A^* z(t) = 0, \quad \text{a.e. } t \in (0,T),$$

that is $A^* z \in \ker \Phi_T^*$. On the other hand, since $z \in D(A^*)$, the map $\Phi_T^* z : t \in [0,T] \mapsto B^* S_A(T-t)^* z \in U$ is continuous and we can take $t = T$ in (21) to obtain that $B^* z = \Phi_T^* z(T) = 0$.

Consequently, the restriction M of A^* to $\ker \Phi_T^*$ is a linear operator from the finite dimensional space $\ker \Phi_T^*$ into itself. Assume that $\ker \Phi_T^* \neq \{0\}$ (otherwise the conclusion (12) is clear from (16)). Therefore, M is triangularizable in $\ker \Phi_T^*$ (here we use that H is a complex Hilbert space). In other words, $\ker \Phi_T^*$ is the direct sum of the root subspaces of M: for every $\lambda \in \sigma(M)$, there exists $m(\lambda) \in \mathbb{N}^*$ such that

$$\ker \Phi_T^* = \bigoplus_{\lambda \in \sigma(M)} \ker (\lambda - M)^{m(\lambda)}.$$

Finally, thanks to (20) we have $\sigma(M) \subset \sigma_F$ and $\ker (\lambda - M)^{m(\lambda)} \subset E_\lambda$ for every $\lambda \in \sigma(M)$. □

Remark 2.8. From the proof of Lemma 2.6 we easily see that the equality (12) remains valid for $T = T_0$ too if $\ker \Phi_{T_0}^* \subset D(A^*)$ (in addition to (11)). Let us also mention an alternative proof in the case $\sigma_F = \emptyset$. Indeed, in this case, the same reasoning as at the end of the proof of Lemma 2.6 easily shows that $\ker \Phi_{T_0}^* \cap D((A^*)^\infty) = \{0\}$. By [22, Proposition 6.1.9] this implies that $\ker \Phi_T^* = \{0\}$ for every $T > T_0$.

Let us now conclude this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. The assumptions of Lemma 2.1 are satisfied for $L = \Phi_T^*$ and $P = \Psi_T^* - \Phi_T^*$ for every $T \geq T_0$. Therefore, for every $T \geq T_0$, we have

$$\dim \ker \Phi_T^* < +\infty, \quad (\ker \Phi_T^*)^\perp = \operatorname{Im} \Phi_T.$$

Applying now Lemma 2.6 we obtain the desired conclusion. □

3. **An example.** Our results, especially Corollary 1.3, potentially have a lot of applications. For instance, it can be used to investigate the exact controllability of some perturbed wave equations (by terms of order zero, as in [24]), plate equations (by terms of order up to one, as in [4]), Schrödinger equations, Korteweg-de Vries equations, transport equations (by non local spatial terms, as in [5]), etc. In this section we focus on a particular example.

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open bounded connected subset with boundary $\partial\Omega$ of class C^2 and let $\omega \subset \Omega$ be a non empty open subset. Let $T > 0$. We consider the following Schrödinger equation with non local term:

$$\begin{cases} y_t = i\Delta y + \displaystyle\int_\Omega k(\xi) y(t,\xi)\, d\xi + \mathbb{1}_\omega(x) u(t,x) & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{22}$$

In (22), y^0 is the initial data, y is the state and u is the control. $\mathbb{1}_\omega$ denotes the function that is equal to 1 in ω and 0 outside. The kernel k is only assumed to be in $L^2(\Omega)$. All the functions we consider in this section are complex-valued functions.

Let us recast (22) as an abstract evolution system (1). The state space H and the control space U are

$$H = U = L^2(\Omega).$$

The operator $A_K : D(A_K) \subset H \longrightarrow H$ is

$$A_K y = i\Delta y + \int_\Omega k(\xi) y(\xi)\, d\xi, \qquad D(A_K) = H^2(\Omega) \cap H_0^1(\Omega),$$

and the control operator $B : U \longrightarrow H$ is

$$Bu = \mathbb{1}_\omega u.$$

Clearly, A_K splits up into $A_K = A_0 + K$, where $A_0 : D(A_0) \subset H \longrightarrow H$ is given by

$$A_0 y = i\Delta y, \qquad D(A_0) = D(A_K),$$

and $K : H \longrightarrow H$ is given by

$$Ky = \int_\Omega k(\xi) y(\xi)\, d\xi.$$

The operator A_0 is skew-adjoint and therefore, by Stone's theorem, it is the generator of a unitary C_0-group on H. On the other hand, it is clear that K is compact. Finally, observe that B is bounded and thus admissible. Therefore, the assumptions (i) and (ii) of Theorem 1.1 are satisfied.

We can now state the following simple (but new) consequence of Corollary 1.3:

Theorem 3.1. *Assume that the Schrödinger equation* (A_0, B) *is exactly controllable in time* $T_0 > 0$*. Let us introduce the subset* $\kappa \subset L^2(\Omega)$ *defined by*

$$\kappa = \left\{ i\Delta\phi + \lambda\phi \;\middle|\; \lambda \in \mathbb{C},\ \phi \in H^2(\Omega) \cap H_0^1(\Omega),\ \phi = 0 \text{ in } \omega,\ \int_\Omega \phi(\xi)\, d\xi = 1 \right\}.$$

If $k \notin \kappa$*, then, for every* $T > T_0$*, (22) is exactly controllable in time* T*. Conversely, if* $k \in \kappa$*, then, for every* $T > 0$*, (22) is not approximately controllable in time* T*.*

Let us point out that κ is never empty, unless $\omega = \Omega$. For explicit conditions of the exact controllability of the Schrödinger equation (A_0, B), we refer to the survey [13] and the references therein. It is for instance nowadays well-known that this equation is exactly controllable in time T_0 for every $T_0 > 0$ if the control domain ω satisfies the so-called Geometric Control Condition.

Proof of Theorem 3.1. Assume first that $k \notin \kappa$. We are going to apply Corollary 1.3. We only have to check the Fattorini-Hautus test corresponding to (22). It is clear that $B^* = B$ and $A_K^* = -A_0 + K^*$ with $D(A_K^*) = D(A_K)$ and

$$K^* z = k \int_\Omega z(\xi)\, d\xi.$$

Let then $\lambda \in \mathbb{C}$ and $z \in H^2(\Omega) \cap H_0^1(\Omega)$ be such that

$$\left\{ \begin{array}{ll} -i\Delta z + k \displaystyle\int_\Omega z(\xi)\, d\xi = \lambda z & \text{in } \Omega, \\ z = 0 & \text{in } \omega, \end{array} \right. \tag{23}$$

and let us show that this implies that $z = 0$ in Ω. Since $k \notin \kappa$, by the very definition of κ, we obtain that the constant $\int_\Omega z(\xi)\, d\xi$ is necessarily equal to zero. Thus, $z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies

$$\left\{ \begin{array}{ll} -i\Delta z = \lambda z & \text{in } \Omega, \\ z = 0 & \text{in } \omega. \end{array} \right.$$

As it is well-known, this implies that $z = 0$ in Ω (we recall that Ω is connected).

Conversely, it is easy to see that, if $k \in \kappa$, then there exists a non zero solution z to (23). □

Remark 3.2. Obviously, we can consider more general kernels $k \in L^2(\Omega \times \Omega)$ that depend on the x variable as well. However, the condition provided by the Fattorini-Hautus test may be less explicit than the one presented in Theorem 3.1. We also chose to apply Corollary 1.3 instead of Theorem 1.1 because it does not seem so easy to check the approximate controllability assumption (iii). One case where it is not difficult to directly check this condition is when the kernel satisfies the analyticity assumption (3) of [8], namely that, for every $y \in L^2(\Omega)$,

$$x \in \Omega \longmapsto \int_\Omega k(\xi, x) y(\xi)\, d\xi \text{ is analytic.} \tag{24}$$

Note that (24) is not satisfied here since we only assumed the kernel to be L^2.

4. **Additional comments.** Let us conclude this article with some final important remarks.

Our main results were obtained as a combination of three ingredients, namely:

1) the desired observability inequality with a "compact error" (6) (obtained in this article as a consequence of the assumptions (i) and (ii)),
2) the Peetre lemma,
3) the general Lemma 2.6.

In the course of this paper we have seen that this procedure is general and we have actually establish the following abstract result:

Theorem 4.1. *Let H and U be two complex Hilbert spaces. Let $A : D(A) \subset H \longrightarrow H$ be the generator of a C_0-semigroup on H and let $B \in \mathcal{L}(U, D(A^*)')$ be admissible for A. For $T \geq 0$ let $\Phi_T \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of (A, B). Assume that there exist a complex Banach space V and a compact operator $P \in \mathcal{L}(H, V)$ such that, for some $T_0 > 0$ and $\alpha > 0$, we have*

$$\alpha \|z\|_H^2 \leq \int_0^{T_0} \left\| \Phi_{T_0}^* z(t) \right\|_U^2 \, dt + \|Pz\|_V^2, \quad \forall z \in H. \tag{25}$$

Then, the set σ_F defined by (9) is finite, the subspace E_λ defined by (10) is finite dimensional for every $\lambda \in \mathbb{C}$, and we have

$$\operatorname{Im} \Phi_T = \left(\bigoplus_{\lambda \in \sigma_F} E_\lambda \right)^\perp, \quad \forall T > T_0.$$

In particular, (A, B) is exactly controllable in time T for every $T > T_0$ if (A, B) satisfies the Fattorini-Hautus test.

In Theorem 1.2, we have provided some conditions on (A, B) to fulfill the assumption (25). However, there are many other techniques that can be used to establish (25). For instance, an easy way to obtain such estimates is to use the so-called multiplier method and then to check to that the "error term" is indeed compact thanks to some compactness results such as the theorem of Aubin-Lions-Simon. This is precisely what is achieved for the plate equation in [15, Théorème IV.3.7] (there, the estimate (3.127) plays the role of (25)) or for a Korteweg-de Vries in [20, Section 3] (see the estimate (3.13)). These are just some examples of equations where the multiplier method leads to (25).

In conclusion, Theorem 4.1 is not only useful to investigate the controllability of compactly perturbed controlled systems, but it is also useful to establish the controllability of numerous unperturbed equations themselves. This abstract result can be considered as a formalization of the compactness-uniqueness method used for the controllability of second order systems in [15] and used to study the stability of some hyperbolic equations in [19].

Remark 4.2. With the notation of Lemma 2.1, we can also show that (a) is equivalent to the following condition:

(c) There exist two subspaces $M, N \subset B_1$ such that

$$B_1 = M \oplus N,$$

with, for some $c > 0$,

$$c \left\|z\right\|_{B_1} \leq \left\|Lz\right\|_{B_2}, \quad \forall z \in M, \tag{26}$$

and $\dim N < +\infty$ with the projection from B_1 onto N being continuous.

It is not difficult to see that (c) implies (a) with $B_3 = B_1$ and P being the projection from B_1 onto N, which is continuous by assumption and compact since its range N is finite dimensional. On the other hand, that (a) implies (c) is actually contained in the proof of Lemma 2.1 (see the proof of ii) $\Longrightarrow$ i) of [18, Lemma 3]).

This provides an equivalent formulation of the assumption (25) which can be useful for some problems. Notably, we see that Theorem 4.1 generalizes [10, Theorem 5.2] by removing some unnecessary spectral assumptions made on the operator A. It is also more general than [14, Lemma 1.1] since this lemma requires that the corresponding right-hand side in (26) defines a norm on B_1, which is equivalent to assuming that (A, B) is approximately controllable in time T_0 (and we recall that this property is stronger than the Fattorini-Hautus test).

Appendix A. **Proof of the estimate (7).** This appendix is devoted to a proof of the estimate (7) that is used in the proof of Lemma 2.3. It is largely inspired by [9, Proposition 3.3].

Let us recall our framework. H and U are two Hilbert spaces. $A : D(A) \subset H \longrightarrow H$ is the generator of a C_0-semigroup $(S_A(t))_{t\geq 0}$ on H and $B \in \mathcal{L}(U, D(A^*)')$ is admissible for A. Let us recall the following dual characterization of admissibility: B is admissible for A if, and only if, for some (and hence all) $T > 0$, there exists $\gamma > 0$ such that

$$\int_0^T \left\|B^* S_A(T-t)^* z\right\|_U^2 \, dt \leq \gamma \left\|z\right\|_H^2, \quad \forall z \in D(A^*). \tag{27}$$

Let us now introduce for $n \in \mathbb{N}$ large enough ($n > \omega_0$, where $\omega_0 \in \mathbb{R}$ is the growth bound of A) the Yosida-like approximations $C_n \in \mathcal{L}(H, U)$ defined by

$$C_n z = n B^* (n - A^*)^{-1} z, \quad \forall z \in H.$$

Let us recall that (see e.g. [6, Lemma II.3.4])

$$n(n - A^*)^{-1} z \xrightarrow[n\to+\infty]{} z \quad \text{in } H, \quad \forall z \in H. \tag{28}$$

This implies in particular that

$$C_n z \xrightarrow[n\to+\infty]{} B^* z \quad \text{in } U, \quad \forall z \in D(A^*), \tag{29}$$

since for every $z \in D(A^*)$ we have

$$\|C_n z - B^* z\|_U \\ \leq \|B^*\|_{\mathcal{L}(D(A^*),U)} \left(\left\|n(n-A^*)^{-1}A^*z - A^*z\right\|_H + \left\|n(n-A^*)^{-1}z - z\right\|_H\right).$$

Let $T > 0$ be fixed. For $f \in L^2(0,T;H)$, let us denote by $S_A^* * f \in L^2(0,T;H)$ the function defined for every $t \in (0,T)$ by

$$(S_A^* * f)(t) = \int_0^t S_A(t-s)^* f(s)\, ds.$$

Using the Cauchy-Schwarz inequality we have

$$\int_0^T \|C_n(S_A^* * f)(t)\|_U^2 \, dt \leq T \int_0^T \int_0^t \left\|B^* S_A(t-s)^* n(n-A^*)^{-1} f(s)\right\|_U^2 \, ds\, dt.$$

Using Fubini's theorem we obtain

$$\int_0^T \|C_n(S_A^* * f)(t)\|_U^2 \, dt \leq T \int_0^T \int_0^T \left\|B^* S_A(t)^* n(n-A^*)^{-1} f(s)\right\|_U^2 \, dt\, ds.$$

Using now the admissibility of B (see (27)) this gives

$$\int_0^T \|C_n(S_A^* * f)(t)\|_U^2 \, dt \leq T\gamma \int_0^T \left\|n(n-A^*)^{-1} f(s)\right\|_H^2 \, ds. \tag{30}$$

Let us now remark that $S_A^* * f \in L^2(0,T;D(A^*))$ for $f \in C^1([0,T];H)$. Using then (29) and (28), we see that the Lebesgue's dominated convergence theorem applies and that we can pass to the limit $n \to +\infty$ in (30) to finally obtain the desired estimate.

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