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INTERNAL CONTROLLABILITY OF FIRST ORDER QUASI-LINEAR HYPERBOLIC SYSTEMS WITH A REDUCED NUMBER OF CONTROLS*

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Abstract. In this paper we investigate the exact controllability of $n \times n$ first order one-dimensional quasi-linear hyperbolic systems by $m < n$ internal controls that are localized in space in some part of the domain. We distinguish two situations. The first one is when the equations of the system have the same speed. In this case, we can use the method of characteristics and obtain a simple and complete characterization for linear systems. Thanks to a linear test this also provides some sufficient conditions for the local exact controllability around the trajectories of semilinear systems. However, when the speed of the equations is not the same, we see that we encounter the problem of loss of derivatives if we try to control quasi-linear systems with a reduced number of controls. To solve this problem, as in a prior article by Coron and Lissy on a Navier–Stokes control system, we first use the notion of algebraic solvability due to Gromov. However, in contrast with this prior article where a standard fixed point argument could be used to treat the nonlinearities, we use here a fixed point theorem of Nash–Moser type due to Gromov in order to handle the problem of loss of derivatives.

Key words. quasi-linear hyperbolic systems, exact internal controllability, controllability of systems, algebraic solvability

AMS subject classifications. 35L50, 93B05, 93C10

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1. Introduction.

In this paper we investigate the exact controllability of $n \times n$ first order one-dimensional quasi-linear hyperbolic systems by $m < n$ internal controls that are localized in space in some part of the domain. While the controllability of quasi-linear hyperbolic systems by boundary controls has been intensively studied, [Cir69, LR02, LR03, Wan06, Zha09, LRW10], to our knowledge there are no equivalent results for the internal controllability. On the other hand, the controllability of systems of PDEs with a reduced number of controls has been a challenging problem for the last decades; see, for instance, [Ala01, AB03] for the first results on linear hyperbolic systems (see also [Dág06, ABL12, AB13, DLRL14, AB14]) and [Zha09] for quasi-linear hyperbolic systems, the survey [AKBGBdT11] (and the references in [dT00], [GBPG05], [Gue07], etc.) for linear parabolic systems, [CGR10] for a nonlinear parabolic system, [CG09b] for Stokes equations, and [FCGIP06], [CG09a], and [CL14] for Navier–Stokes equations. Let us also point out that, in many of these articles, the general strategy is to start with a controllability result in the case where there are as many controls as the number of equations and then to try to remove some of these controls by a suitable procedure. In the present article we will follow this general strategy by making use of the so-called fictitious control method introduced in [Cor92] for the control of linear ODEs and in [GBPG05] for the control of linear partial differential equations, an article where this illuminating terminology was introduced.

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In [LR03], the authors introduced a constructive method to control quasi-linear systems of n equations by n boundary controls. This proficient method is based on existence and uniqueness results of semiglobal solutions [LJ01] (i.e., with large time and small data) that they apply to several mixed initial-boundary value problems, using also the equivalent roles of the time and the space. As we shall see below, using a method of extension of the domain (as it is often used in the parabolic framework), we can recover this result for the internal controllability, that is, we can prove the controllability of $n \times n$ quasi-linear systems by n internal controls. The situation is more complicated when we have fewer controls than equations. Indeed the extension method is not applicable in this context. Thus, we need to develop direct methods to solve the problem of internal controllability.

We start the study with linear systems of equations with the same velocity. In this case, we can apply the method of characteristics and obtain a complete and simple characterization of the exact controllability. We show that the linear system can be viewed as a parameterized family of ODEs that are controlled independently. The difficulty is actually to prove that this is enough to build a smooth (C^1) control for the linear hyperbolic system. Moreover, since we look for controls of the hyperbolic system that are localized in some part of the domain, a nonstandard condition on the supports of the ODEs also appears and needs to be handled. Another key point of the proof is the explicit formula of the Hilbert uniqueness method control for ODEs.

Using then a standard fixed point argument we can obtain sufficient conditions for the local exact controllability around the trajectories of semilinear systems. However, when the equations do not have the same speed anymore and the nonlinearity is stronger, that is, when we consider quasi-linear systems, the standard linear test fails because of a loss of derivatives. To solve this problem, we need to use a fixed point of Nash–Moser type. We propose to use the fixed point theorem of Gromov [Gro86, section 2.3.2, Main Theorem], which is based on the notion of algebraic solvability for partial differential operators (see Definition 3.4 below for more details). The method consists in first controlling the $n \times n$ system by n controls, and then eliminating a certain number of controls through the algebraic solvability. The use of the Gromov algebraic solvability in the framework of the control theory was introduced in [Cor07, pp. 13–15] for the control of linear ODEs (however, it does not lead to new results in this case), in [CL14] for a Navier–Stokes control system, and in [DL16a, DL16b] for some first order coupled parabolic systems. In these works, the parabolicity allows us to have smooth controls and thus to avoid the problem of loss of derivatives. The difference between the present work and [CL14], where the algebraic solvability was the difficult task (the fixed point was standard), is that, following the algebraic solvability step, we show how to apply the fixed point theorem of Gromov to obtain the controllability of the quasi-linear system. Last, but not least, this method is probably not optimal with the regularity obtained, which leaves some challenging problems.

2. Systems of equations with the same velocity.

2.1. Linear systems. Let us consider the following linear hyperbolic system with periodic boundary conditions:

$$
\begin{cases} y_t + y_x + A(t,x)y = B(t,x)\Theta, \quad (t,x) \in [0,T] \times [0,L], \\ y(t,L) = y(t,0), \quad t \in [0,T], \\ y(0,x) = y^0(x), \quad x \in [0,L]. \end{cases} \tag{2.1}
$$

In (2.1), $T > 0$ is the control time, and $L > 0$ is the length of the domain. A and B are time and space dependent matrices of size $n \times n$ and $n \times m$, respectively, where $n \in \mathbb{N}^*$ denotes the number of equations of the system and $m \in \mathbb{N}^*$ the number of controls (with possibly $m < n$). y^0 is the initial data and $y(t, \cdot) : [0, L] \longrightarrow \mathbb{R}^n$ is the state at time $t \in [0, T]$. Finally, $\Theta(t, \cdot) : [0, L] \longrightarrow \mathbb{R}^m$ is the distributed control at time $t \in [0, T]$, subject to the constraint

$$\text{supp}\Theta \subset [0, T] \times [a, b]; \tag{2.2}$$

here, and in what follows, the interval $[a, b]$, with $0 \leq a < b \leq L$, is fixed.

Throughout this article, for $k \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we denote by $C^k_L([0, T] \times [0, L])^p$ (resp., $C^k_L([0, L])^p$) the Banach space of functions $y \in C^k([0, T] \times [0, L])^p$ (resp., $y \in C^k([0, L])^p$) that are L-periodic w.r.t. x, that is,

$$\partial^i_x y(t, 0) = \partial^i_x y(t, L) \quad \forall t \in [0, T], \quad \forall i \in [\![0, k]\!], \tag{2.3a}$$

$$\left(\text{resp., } y^{(i)}(0) = y^{(i)}(L), \quad \forall i \in [\![0, k]\!]\right). \tag{2.3b}$$

Throughout section 2.1 we assume that $A \in C^1_L([0, T] \times [0, L])^{n \times n}$, $B \in C^1_L([0, T] \times [0, L])^{n \times m}$. These assumptions are made for regularity purposes; see below.

We recall that, for every $T > 0$, there exists $C > 0$ such that, for every $\Theta \in C^1_L([0, T] \times [0, L])^m$ and every $y^0 \in C^1_L([0, L])^n$, there exists a unique classical global solution $y \in C^1_L([0, T] \times [0, L])^n$ to (2.1), and this solution satisfies the estimate

$$\|y\|_{C^1} \leq C \left(\|y^0\|_{C^1} + \|\Theta\|_{C^1}\right).$$

This well-posedness result follows from the classical theory of linear hyperbolic systems using the method of characteristics [LY85]. Note that the kind of boundary conditions we consider are nonlocal but, as already noticed in [CBdN08] (see also [LRW10]), they can always be reduced to more standard (i.e., local) boundary conditions by introducing the enlarged system satisfied by $(y, \tilde{y})$, where $\tilde{y}(t, x) = y(t, L - x)$.

Definition 2.1. *We say that system* (2.1) *is exactly controllable at time* $T > 0$ *if, for every* $y^0 \in C^1_L([0, L])^n$ *and for every* $y^1 \in C^1_L([0, L])^n$*, there exists a control* $\Theta \in C^1_L([0, T] \times [0, L])^m$ *that satisfies the constraint* (2.2) *and is such that the corresponding solution* $y \in C^1_L([0, T] \times [0, L])^n$ *to* (2.1) *satisfies*

$$y(T, x) = y^1(x) \quad \forall x \in [0, L].$$

2.1.1. The extended characteristics. Let us now introduce an important tool when dealing with hyperbolic systems, namely, the characteristics of the system. In our case (speed 1 on each equation), the characteristic X of system (2.1) passing through the point $(t_0, x_0) \in [0, T] \times [0, L]$ is the straight line

$$X(t, t_0, x_0) \stackrel{\text{def}}{=} t - t_0 + x_0, \quad t \in [0, T].$$

However, in this paper, the crucial tools we need are the *extended characteristics* $\overline{X} : [0, T] \times [0, L) \longrightarrow [0, L]$, defined by (see Figure 1 below)

$$\overline{X}(t, x) \stackrel{\text{def}}{=} \begin{cases} X(t, 0, x) & \text{if } t \in [0, \tau_0(x, L)], \\ X(t, \tau_{k-1}(x, L), 0) & \text{if } t \in (\tau_k(x, 0), \tau_k(x, L)],\ k \in [\![k_{\min}(x, 0), k_{\max}(x, 0)]\!], \end{cases}$$

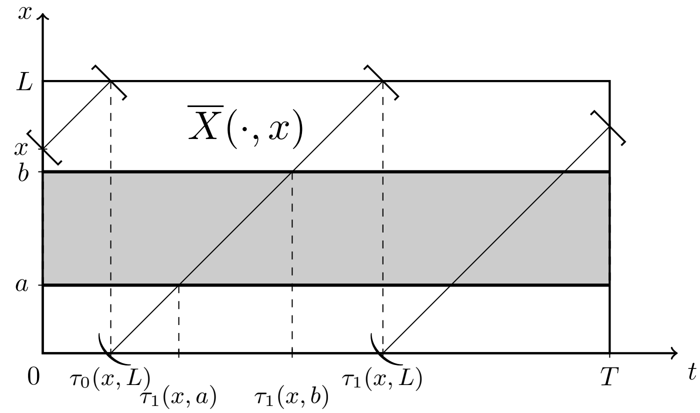

FIG. 1. *The extended characteristics* $\overline{X}(\cdot,x)$.

where, for every $k\in\mathbb{N}$ and $c\in[0,L]$, we introduce the functions

$$\tau_k(x,c)\overset{\text{def}}{=}\begin{cases}0 & \text{if } c-x+kL\in(-\infty,0),\\ c-x+kL & \text{if } c-x+kL\in[0,T],\\ T & \text{if } c-x+kL\in(T,+\infty),\end{cases}$$

and $k_{\min}(x,c)$ (resp., $k_{\max}(x,c)$) denotes the smallest (resp., greatest) integer $k\in\mathbb{N}$ such that $c-x+kL>0$ (resp., $c-x+kL<T$). More precisely, denoting by $\lfloor\cdot\rfloor$ the floor function and $\lceil\cdot\rceil$ the ceiling function,

$$k_{\min}(x,c)\overset{\text{def}}{=}\lfloor\frac{-c+x}{L}\rfloor+1=\begin{cases}0 & \text{if } x\in[0,c),\\ 1 & \text{if } x\in[c,L),\end{cases}$$

$$k_{\max}(x,c)\overset{\text{def}}{=}\lceil\frac{T-c+x}{L}\rceil-1=\begin{cases}\lceil\frac{T-c}{L}\rceil-1 & \text{if } x\in[0,p(c)]\,,\\ \lceil\frac{T-c}{L}\rceil & \text{if } x\in(p(c),L)\,,\end{cases}$$

where

$$p(c)\overset{\text{def}}{=}\left(\lceil\frac{T-c}{L}\rceil-\frac{T-c}{L}\right)L.$$

Note that $\tau_k(x,0)=\tau_{k-1}(x,L)$ for every $k\geq 1$ and $\tau_{k_{\max}(x,0)}(x,L)=T$, so that $\overline{X}(t,x)$ is indeed defined for every $t\in[0,T]$.

For $0\leq a<b\leq L$, we list below some properties of these functions, under the essential assumption that every extended characteristic $\overline{X}$ crosses the domain $[0,T]\times[a,b]$ at some time, that is,

$$T>L-(b-a).$$

1. $k_{\min}(x,b)\leq k_{\max}(x,a)$.
2. $\tau_k(x,a)<\tau_k(x,b)$ for every $k\in[\![k_{\min}(x,b),k_{\max}(x,a)]\!]$ if $x\neq b$ and $x\neq p(a)$.
3. $\tau_k(x,b)\leq\tau_{k+1}(x,a)$ for every $k\in[\![k_{\min}(x,b),k_{\max}(x,a)-1]\!]$.

4. $\overline{X}(t,x) \in (a,b)$ for every $t \in (\tau_k(x,a), \tau_k(x,b))$ for every k satisfying

$$k \in [\![k_{\min}(x,b), k_{\max}(x,a)]\!].$$

We then introduce the open sets (see Figure 2 below)

$$\mathcal{T}_0 \overset{\text{def}}{=} \{(t,x) \in (0,T) \times (0,L) \quad | \quad t \in (0, \tau_0(x,L))\},$$

$$\mathcal{T}_{\lceil \frac{T}{L} \rceil} \overset{\text{def}}{=} \left\{(t,x) \in (0,T) \times (p(0),L), \quad \middle| \quad t \in \left(\tau_{\lceil \frac{T}{L} \rceil}(x,0), T\right)\right\},$$

and, for $k \in [\![1, \lceil \frac{T}{L} \rceil - 1]\!]$,

$$\mathcal{T}_k \overset{\text{def}}{=} \{(t,x) \in (0,T) \times (0,L) \quad | \quad t \in (\tau_k(x,0), \tau_k(x,L))\}.$$

We set

$$\mathcal{T} \overset{\text{def}}{=} \bigcup_{k=0}^{\lceil \frac{T}{L} \rceil} \mathcal{T}_k.$$

Remark 1. Let us give some comments about the properties of the extended characteristics. First, we have

$$\overline{X} \in C^1(\mathcal{T})$$

with $\overline{X}_t(t,x) = \overline{X}_x(t,x) = 1$ for every $(t,x) \in \mathcal{T}$. Moreover, at the boundary $\partial\mathcal{T}$, we have

$$\forall (t_0,x_0) \in \overline{\mathcal{T}_k} \cap \overline{\mathcal{T}}_{k+1}, \qquad \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\mathcal{T}_k}} \overline{X}(t,x) = L, \qquad \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\mathcal{T}_{k+1}}} \overline{X}(t,x) = 0,$$

$\lim_{(t,x)\to(t_0,0)} \overline{X}(t,x) = \overline{X}(t_0,0)$ for every $t_0 \in [0,T]$, and

$$\forall t_0 \in [0,T], \qquad \lim_{(t,x)\to(t_0,L)} \overline{X}(t,x) = \begin{cases} \overline{X}(t_0,0) & \text{if } t_0 \in (0,T], \\ L & \text{if } t_0 = 0. \end{cases}$$

In particular $\overline{X}$ has a continuous extension (still denoted by $\overline{X}$) to all the points of the boundary (t,L), $t \in [0,T]$.

Second, the map $(t,x) \mapsto (t, \overline{X}(t,x))$ is a C^1-diffeomorphism from $\mathcal{T}$ to $\mathcal{T}' = (0,T) \times (0,L) \setminus \left\{(t, \overline{X}(t,0)) \,\middle|\, t \in [0,T]\right\}$. We denote by $(t, \overline{X}^{-1}(t,x))$ its inverse.

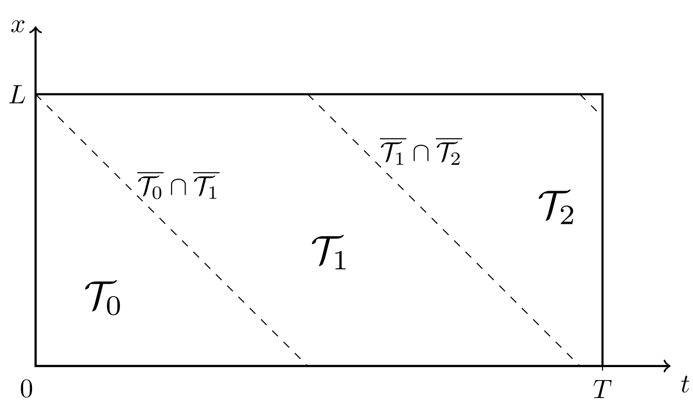

FIG. 2. *Domains $\mathcal{T}_k$ and the parts $\overline{\mathcal{T}_k} \cap \overline{\mathcal{T}}_{k+1}$ of their boundary.*

Finally, for every $x \in [0, L]$, we denote by A^x and B^x the values of A and B along the extended characteristic $\overline{X}(\cdot, x)$:

$$A^x(t) \stackrel{\text{def}}{=} A(t, \overline{X}(t,x)), \quad B^x(t) \stackrel{\text{def}}{=} B(t, \overline{X}(t,x)) \quad \forall t \in [0,T].$$

Clearly, $(t,x) \mapsto A^x(t) \in C^1(\mathcal{T})^{n\times n}$. Since A is L-periodic w.r.t. x, we have $A \in C^0\left(\overline{\mathcal{T}}\right)^{n\times n} = C^0([0,T]\times[0,L])^{n\times n}$. In the same manner, since $\partial_t A$ and $\partial_x A$ are L-periodic w.r.t. x, we have $A \in C^1([0,T]\times[0,L])^{n\times n}$. Note also that $A_L = A_0$. The same statements hold for the map $(t,x) \mapsto B^x(t)$ as well.

2.1.2. Characterization of the controllability of (2.1). The main result of section 2.1 is the following.

THEOREM 2.2. *Let $T, L > 0$ and $0 \le a < b \le L$. System* (2.1) *is exactly controllable at time T if and only if the following two conditions hold:*
($\mathcal{H}_1$) $T > L - (b-a)$.
($\mathcal{H}_2$) *For every $x \in [0, L)$, the following ODE is controllable:*

$$\begin{cases} \dfrac{d}{dt} z(t) = -A^x(t) z(t) + B^x(t)\psi(t) & \forall t \in [0,T], \\ z(0) = z^0 \in \mathbb{R}^n, \end{cases} \tag{2.4}$$

with controls $\psi \in C^1([0,T])^m$ such that

$$\psi \equiv 0 \ \text{in} \ [0,T] \backslash \left(\bigcup_{k=k_{\min}(x,b)}^{k_{\max}(x,a)} [\tau_k(x,a), \tau_k(x,b)] \right). \tag{2.5}$$

Remark 2. When $[a,b] = [0,L]$, hypothesis ($\mathcal{H}_1$) and (2.5) are automatically satisfied.

Remark 3. As we shall see below (Proposition 2.3) the controllability of (2.4) with (2.5) only depends on the values of A and B inside the control domain $[0,T]\times[a,b]$.

Remark 4. In the proof of Theorem 2.2 we will explicitly construct a control Θ that steers the solution y to (2.1) from y^0 to y^1; see (2.14) below. We can see that this control Θ satisfies the following additional properties:

1. Continuity: there exists $C > 0$ (depending only on T, L, a, b, A, B) such that

$$\|\Theta\|_{C^1} \le C \left(\|y^0\|_{C^1} + \|y^1\|_{C^1} \right).$$

2. Locality: there exists $\delta > 0$ small enough (depending only on T, L, a, b, A, B), such that

$$\text{supp}\Theta \subset [\delta, T-\delta] \times [a+\delta, b-\delta]. \tag{2.6}$$

3. Higher regularity: if $y^0, y^1 \in C^k_L([0,L])^n$ and $A \in C^k_L([0,T]\times[0,L])^{n\times n}$, $B \in C^k_L([0,T]\times[0,L])^{m\times n}$ ($k \ge 1$), then

$$\Theta \in C^k([0,T]\times[0,L])^m.$$

2.1.3. Controllability of linear ODE with constraints. Let us recall that we know some powerful tools to characterize the controllability of linear time-varying ODEs if no constraints are imposed on the controls. We state below the extensions of these theorems to the case where the controls are supported in some part of the domain.

Let us consider the $n \times n$ ODE

$$\begin{cases} \dfrac{d}{dt} z(t) = -A(t)z(t) + B(t)\psi(t) \quad \forall t \in [0,T], \\ z(0) = z^0 \in \mathbb{R}^n, \end{cases} \tag{2.7}$$

with $A \in C^1([0,T])^{n\times n}$, $B \in C^1([0,T])^{n\times m}$. We want to characterize the controllability of (2.7) with the following additional constraint on the controls:

$$\psi \equiv 0 \text{ in } [0,T] \backslash \left(\bigcup_{i=1}^{M} [a_i, b_i] \right), \tag{2.8}$$

where $0 \le a_i < b_i \le T$ are such that $b_i \le a_{i+1}$ for every $i \in [\![1, M-1]\!]$.

Let us denote by $R \in C^1([0,T] \times [0,T])^{n\times n}$ the resolvent associated with $-A \in C^1([0,T])^{n\times n}$, that is, for every $s \in [0,T]$, $R(\cdot, s)$ is the classical solution to the ODE

$$\begin{cases} \partial_t R(t,s) = -A(t)R(t,s) \quad \forall t \in [0,T], \\ R(s,s) = \mathrm{Id}. \end{cases}$$

Proposition 2.3. *The ODE* (2.7) *is controllable with* (2.8) *if and only if its controllability Gramian, that is, the $n \times n$ matrix*

$$Q \stackrel{\text{def}}{=} \sum_{i=1}^{M} \int_{a_i}^{b_i} R(T,s)\, B(s)B(s)^* R(T,s)^* \, ds, \tag{2.9}$$

is invertible.

The proof of Proposition 2.3 can be adapted from that of [KHN63, Theorem 5]. To do so, we consider the control problem (2.7) with ηB instead of B, where η is a cut-off function that vanishes outside $\bigcup_{i=1}^{M} [a_i, b_i]$ and is equal to 1 in $\bigcup_{i=1}^{M} [a_i + \varepsilon, b_i - \varepsilon]$ with $\varepsilon > 0$ small enough so that, by continuity, the Gramian (2.9) with $a_i + \varepsilon$ (resp., $b_i - \varepsilon$) instead of a_i (resp., b_i) remains invertible.

Thanks to the previous characterization, we obtain the following proposition (see [KHN63, Theorem 10] and [SM67] for a proof).

Proposition 2.4. *Assume that A and B are constant. Then, the controllability of* (2.7)–(2.8) *is equivalent to the algebraic condition*

$$\operatorname{rank}[\, A : B \,] = n,$$

where the $n \times nm$ matrix $[\, A : B \,]$ is defined by

$$[\, A : B \,] \stackrel{\text{def}}{=} [B|AB|A^2B|\cdots|A^{n-1}B]. \tag{2.10}$$

Proposition 2.5. *Assume that*

$$A \in C^{n-2}([0,T])^{n\times n} \text{ and } B \in C^{n-1}([0,T])^{n\times m}$$

and let us introduce the following notation:

$$\forall t \in [0,T], \quad \begin{cases} B_0(t) = B(t), \\ B_j(t) = \dfrac{d}{dt} B_{j-1}(t) + A(t)B_{j-1}(t) \quad \forall j \in [\![1, n-1]\!], \end{cases}$$

and, for every $t \in [0,T]$, the $n \times nm$ matrix

$$[\,A\,:\,B\,]\,(t) \stackrel{\text{def}}{=} [B_0(t)|B_1(t)|\cdots|B_{n-1}(t)]$$

(which provides an extension of (2.10)*). Then, the ODE* (2.7) *is controllable with* (2.8) *if the following property holds:*

$$\exists i \in [\![1, M]\!], \quad \exists t_i \in [a_i, b_i], \quad \operatorname{rank}[\,A\,:\,B\,]\,(t_i) = n.$$

2.1.4. Proof of Theorem 2.2, sufficient part. The proof of the sufficient part of Theorem 2.2 relies on the following key lemma (the proof of which is postponed to Appendix A). It states that we can always reduce a little bit the domain of control. This is a uniform result with respect to x (compare with Proposition 2.3).

Throughout this section, for $x \in [0,L]$ we denote by $R^x \in C^1([0,T]\times[0,T])^{n\times n}$ the resolvent associated with $-A^x \in C^1([0,T])^{n\times n}$. It is important to notice that the map $(t,s,x) \mapsto R^x(t,s)$ is of class $C^1([0,T]\times[0,T]\times[0,L])^{n\times n}$ since the map $(t,x)\mapsto A^x(t)$ is also of class $C^1([0,T]\times[0,L])^{n\times n}$ (see, for instance [Har82, Chapter V, Theorem 3.1]).

Lemma 2.6. *Assume that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Then, there exists* $\delta > 0$ *small enough and a cut-off function* $\eta \in C^1([0,T]\times[0,L])$ *with (see Figure* 3*)*

$$\eta \equiv 0 \text{ in } [0,T]\times[0,L]\backslash\left((\delta, T-\delta)\times(a+\delta, b-\delta)\right), \tag{2.11}$$

such that, for every $x \in [0,L]$, the Gramian

$$Q_x \stackrel{\text{def}}{=} \int_0^T R^x\,(T,s)\,B^x(s)B^x(s)^* R^x\,(T,s)^*\,\eta(s,\overline{X}(s,x))\,ds \tag{2.12}$$

is invertible.

Proof of Theorem 2.2 (sufficient part). Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold and let us show that system (2.1) is exactly controllable at time T. Let $y^0, y^1 \in C^1_L([0,L])^n$.

Let Q_x be the controllability Gramian defined by (2.12). For every $(t,x)\in[0,T]\times[0,L]$, we set

$$\psi(t,x) \stackrel{\text{def}}{=} \eta(t,\overline{X}(t,x))B^x(t)^*R^x(T,t)^*Q_x^{-1}\left(y^1\left(\overline{X}(T,x)\right) - R^x(T,0)y^0(x)\right). \tag{2.13}$$

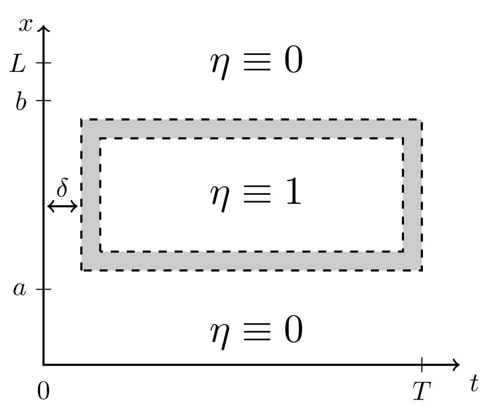

Fig. 3. *Reduction of the control domain.*

Since $\eta \in C^1_L([0,T]\times[0,L])$, we have $(t,x)\mapsto \eta(t,\overline{X}(t,x)) \in C^1([0,T]\times[0,L])$. Using Lebesgue's dominated convergence theorem, we obtain $x\mapsto Q_x \in C^1([0,L])^{n\times n}$. On the other hand, since $y^1\in C^1_L([0,L])^n$, we have $x\mapsto y^1(\overline{X}(T,x))\in C^1([0,L])^n$. As a result,

$$\psi \in C^1([0,T]\times[0,L])^m.$$

For every $(t,x)\in[0,T]\times[0,L]$, we set

$$\Theta(t,x) \overset{\text{def}}{=} \begin{cases} \psi(t,\overline{X}^{-1}(t,x)) & \text{if } (t,x)\in\mathcal{T}', \\ \psi(t,0) & \text{if } x=\overline{X}(t,0), \\ 0 & \text{if } (t,x)\in\partial\left([0,T]\times[0,L]\right). \end{cases} \tag{2.14}$$

From (2.11) we have $\Theta\in C^1([0,T]\times[0,L])^m$ and (2.6). Moreover,

$$\Theta(t,\overline{X}(t,x)) = \psi(t,x) \quad \forall (t,x)\in[0,T]\times[0,L].$$

Let $y\in C^1([0,T]\times[0,L])^m$ be the solution to (2.1) associated with Θ defined by (2.14). For every $x\in[0,L)$, writing y along the extended characteristics $\overline{X}(\cdot,x)$, we see that $t\mapsto y(t,\overline{X}(t,x))$ solves the ODE (2.4) with ψ defined by (2.13) and $z^0=y^0(x)$ (at least in the weak sense $W^{1,\infty}(0,T)^n$). As a result we obtain $y(T,\overline{X}(T,x)) = y^1(\overline{X}(T,x))$ for every $x\in[0,L)$. Since $x\mapsto\overline{X}(T,x)$ defines a bijective map from $[0,L)$ to $(0,L]$, we obtain that $y(T,x)=y^1(x)$ for every $x\in[0,L)$. By continuity it follows that $y(T,x)=y^1(x)$ for every $x\in[0,L]$. ∎

2.1.5. Proof of Theorem 2.2, necessary part. Assume now that system (2.1) is exactly controllable at time T and let us prove that this implies that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold.

Assume first that $0<T\le L-(b-a)$ and let $t_0=\max(0,T-a)$ and $x_0=-t_0+L$. Note that $t_0\in[0,T]$ and $x_0\in[0,L]$. Let $y^1=0$ and let $y^0\in C^1_L([0,L])^n$ be such that

$$y^0(x_0)\neq 0. \tag{2.15}$$

Writing y along the characteristic $X(s,t_0,0)$ for $s\in[t_0,T]$, gives

$$y(T,X(T,t_0,0)) = R_1(T,t_0)y(t_0,0) + \int_{t_0}^T R_1(T,s)B(s)\Theta(s,X(s,t_0,0))\,ds,$$

where R_1 is the resolvent associated with $t\in[t_0,T]\mapsto A(t,X(t,t_0,0))$. Now observe that, since $T\le t_0+a$, we have $X(s,t_0,0)\le a$ for $s\in[t_0,T]$, so that, thanks to (2.2),

$$\Theta(s,X(s,t_0,0))=0 \quad \forall s\in[t_0,T].$$

As a result,

$$y(T,X(T,t_0,0)) = R_1(T,t_0)y(t_0,0). \tag{2.16}$$

Similarly, writing y along the characteristic $X(s,0,x_0)$ for $s\in[0,t_0]$, and using that $x_0\ge b$, this leads to

$$y(t_0,L) = y(t_0,X(t_0,0,x_0)) = R_2(t_0,0)y^0(x_0),$$

where R_2 is the resolvent associated with $t\in[0,t_0]\mapsto A(t,X(t,0,x_0))$. Since $y(t_0,0)=y(t_0,L)$, the previous equality can be combined with (2.16) and (2.15) to show that $y(T, X(T, t_0, 0)) \neq 0$ and therefore system (2.1) is not exactly controllable at time T.

We turn to the necessity of $(\mathcal{H}_2)$. Let $z^0, z^1 \in \mathbb{R}^n$ be fixed. We then define $y^0 \equiv z^0$ and $y^1 \equiv z^1$, which belong to $C^1_L([0, L])^n$. Thus, by assumption, there exists $\Theta \in C^1_L([0, T] \times [0, L])^m$ that satisfies (2.2) such that the corresponding solution $y \in C^1([0, T] \times [0, L])^n$ to (2.1) satisfies

$$y(T, x) = z^1 \quad \forall x \in [0, L]. \tag{2.17}$$

For every $(t, x) \in [0, T] \times [0, L)$, we set

$$\psi(t) \stackrel{\text{def}}{=} \Theta\left(t, \overline{X}(t, x)\right). \tag{2.18}$$

Since $\Theta \in C^1_L([0, T] \times [0, L])^m$, we have $\psi \in C^1([0, T])^m$, while (2.5) follows from (2.2). Let $z \in C^1([0, T])^n$ be the solution to (2.4) associated with ψ defined by (2.18). Let $x \in [0, L)$ being fixed. Writing y along the extended characteristic $\overline{X}(\cdot, x)$, we see that $t \in [0, T] \mapsto y(t, \overline{X}(t, x))$ solves (2.4) with $z^0 = y^0(x)$ (at least in the weak sense $W^{1,\infty}(0, T)^n$). By uniqueness of the solution to (2.4) we then have

$$z(t) = y\left(t, \overline{X}(t, x)\right) \quad \forall t \in [0, T].$$

In particular, $z(T) = z^1$ thanks to (2.17).

2.2. Semilinear systems. Let us consider the following semilinear first order hyperbolic system with periodic boundary conditions:

$$\begin{cases} y_t + y_x = f(y, \Theta), & (t, x) \in [0, T] \times [0, L], \\ y(t, L) = y(t, 0), & t \in [0, T], \\ y(0, x) = y^0(x), & x \in [0, L]. \end{cases} \tag{2.19}$$

We assume that the nonlinearity $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is of class C^2.

Definition 2.7. *We say that* $(\tilde{y}, \tilde{\Theta}) \in C^1_L([0, T] \times [0, L])^n \times C^1_L([0, T] \times [0, L])^m$ *is a trajectory of system* (2.19) *if it is a classical solution to* (2.19) *for some* $y^0 \in C^1_L([0, L])^n$ *and if* $\tilde{\Theta}$ *satisfies* (2.2).

Definition 2.8. *Let* $(\tilde{y}, \tilde{\Theta})$ *be trajectory of system* (2.19). *We say that system* (2.19) *is locally exactly controllable around the trajectory* $(\tilde{y}, \tilde{\Theta})$ *at time* $T > 0$ *if, for every* $\varepsilon > 0$, *there exists* $\mu > 0$ *such that, for every* $y^0, y^1 \in C^1_L([0, T] \times [0, L])^n$ *with*

$$\|y^0 - \tilde{y}(0, \cdot)\|_{C^1} \leq \mu, \quad \|y^1 - \tilde{y}(T, \cdot)\|_{C^1} \leq \mu,$$

there exists a control $\Theta \in C^1_L([0, T] \times [0, L])^m$ *that satisfies* (2.2) *and a classical solution* $y \in C^1_L([0, T] \times [0, L])^n$ *to* (2.19) *such that*

$$y(T, x) = y^1(x) \quad \forall x \in [0, L], \tag{2.20a}$$

$$\|y - \tilde{y}\|_{C^1} \leq \varepsilon, \tag{2.20b}$$

$$\|\Theta - \tilde{\Theta}\|_{C^1} \leq \varepsilon. \tag{2.20c}$$

Then, we have the following result. The proof is classical and use the Banach fixed point theorem (see, for instance, [Cor07, section 4.1]).

THEOREM 2.9. *Let* $(\tilde{y}, \tilde{\Theta})$ *be a trajectory of system* (2.19). *Assume that the linearization of system* (2.19) *around the trajectory* $(\tilde{y}, \tilde{\Theta})$*, that is the linear system*

$$\begin{cases} y_t + y_x = \dfrac{\partial f}{\partial y}\left(\tilde{y}(t,x), \tilde{\Theta}(t,x)\right) y + \dfrac{\partial f}{\partial \Theta}\left(\tilde{y}(t,x), \tilde{\Theta}(t,x)\right)\Theta, \ (t,x) \in [0,T] \times [0,L], \\ y(t,L) = y(t,0), \quad t \in [0,T], \\ y(0,x) = y^0(x), \quad x \in [0,L], \end{cases}$$

is exactly controllable at time $T > 0$. *Then, system* (2.19) *is locally exactly controllable around the trajectory* $(\tilde{y}, \tilde{\Theta})$ *at time* T.

3. Quasi-linear systems with different velocities. In what follows, we denote by $e_1 = (1,0)$, $e_2 = (0,1)$ the canonical basis of $\mathbb{R}^2$.

We are now interested in the controllability of the following 2×2 quasi-linear system by one control force

$$\begin{cases} y_t + \Lambda(y) y_x + f(y) = e_1 \Theta, \quad (t,x) \in [0,T] \times [0,L], \\ y(t,L) = y(t,0), \quad t \in [0,T], \\ y(0,x) = y^0(x), \quad x \in [0,L], \end{cases} \tag{3.1}$$

where

$$\Lambda(y) = \operatorname{diag}(\lambda_1(y), \lambda_2(y)) \quad \forall y \in \mathbb{R}^2$$

with

$$\lambda_1(y) < \lambda_2(y), \quad \lambda_1(y) \neq 0, \quad \lambda_2(y) \neq 0 \quad \forall y \in \mathbb{R}^2 \tag{3.2}$$

and

$$f(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} \quad \forall y \in \mathbb{R}^2$$

with

$$f_1(0) = f_2(0) = 0,$$

so that $(0,0)$ is a trajectory of system (3.1). We assume that $\lambda_1, \lambda_2, f_1, f_2 \in C^\infty(\mathbb{R}^2)$. In particular, system (3.1) is hyperbolic (see, for instance, [LY85, pp. 1–2]).

Then, for every $T > 0$, there exist $C > 0$ and $\mu > 0$ such that, for every $\Theta \in C^k_L([0,T] \times [0,L])$ and every $y^0 \in C^k_L([0,T] \times [0,L])^2$ ($k \in \mathbb{N}^*$) such that

$$\|\Theta\|_{C^k} \leq \mu, \quad \|y^0\|_{C^k} \leq \mu,$$

there exists a unique semiglobal classical solution $y \in C^k_L([0,T] \times [0,L])^2$ to (3.1), and this solution satisfies the estimate

$$\|y\|_{C^k} \leq C\left(\|y^0\|_{C^k} + \|\Theta\|_{C^k}\right).$$

We refer to [LJ01], [Wan06] for a proof of this well-posedness result.

The technical point in the method we will develop lies in the algebraic solvability (see section 3.2 below). Since the eigenvalues of $\Lambda(y)$ might be distinct, the more the number n of equations of the system is large, the more it becomes difficult to solve algebraically the system. That is why we restrict ourselves to the case of $n = 2$

equations. We also see during this step that we have to take the derivatives of the coefficients of the equations, which shows the loss of derivatives. When $n > 2$, the algebraic solvability becomes a difficult task that involves the same arguments as in [CL14] to be solved. This is not the purpose of the present paper but this could be the investigation of further developments. On the other hand, once the algebraic solvability is established (under some conditions), the rest of the proof of Theorem 3.1 below remains unchanged whether $n = 2, 3, \ldots$.

Our main result is the following local exact controllability result around the trajectory $(0, 0)$.

THEOREM 3.1. *Assume that*

$$T > (L - (b - a)) \max \left\{ \frac{1}{|\lambda_1(0)|}, \frac{1}{|\lambda_2(0)|} \right\} \tag{3.3}$$

and

$$\frac{\partial f_2}{\partial y_1}(0) \neq 0. \tag{3.4}$$

Then, for every $\varepsilon > 0$, *there exists* $\mu > 0$ *such that, for every* $y^0, y^1 \in C^6_L([0, L])^2$ *that satisfy*

$$\|y^0\|_{C^6} \leq \mu, \quad \|y^1\|_{C^6} \leq \mu,$$

there exists a control $\Theta \in C^1_L([0, T] \times [0, L])$ *that satisfies*

$$\mathrm{supp}\Theta \subset [\delta, T - \delta] \times [a + \delta, b - \delta], \tag{3.5a}$$

$$\|\Theta\|_{C^1} \leq \varepsilon, \tag{3.5b}$$

for every $0 < \delta < \min(T, (b - a)/2)/4$ *such that*

$$T - 4\delta > (L - (b - a - 8\delta)) \max \left\{ \frac{1}{|\lambda_1(0)|}, \frac{1}{|\lambda_2(0)|} \right\} \tag{3.6}$$

and such that the corresponding solution $y \in C^1_L([0, T] \times [0, L])^n$ *to* (3.1) *satisfies*

$$y(T, x) = y^1(x) \quad \forall x \in [0, L], \tag{3.7a}$$

$$\|y\|_{C^1} \leq \varepsilon. \tag{3.7b}$$

Remark 5. Recall that, given C^1 data y^0 and Θ, (3.1) has a C^1 solution. Now observe that in Theorem 3.1 the initial and final data are smoother than the control. This gap between the regularities is not a technical matter (even though the regularity C^6 can probably be weakened). Indeed, consider the linear system

$$\begin{cases} u_t + u_x = \Theta, \quad (t, x) \in [0, T] \times [0, L], \\ v_t + \lambda v_x + u = 0, \quad (t, x) \in [0, T] \times [0, L], \\ u(t, L) = u(t, 0), \quad v(t, L) = v(t, 0), \quad t \in [0, T], \\ u(0, x) = u^0(x), \quad v(0, x) = v^0(x), \quad x \in [0, L], \end{cases}$$

with $\lambda > 1$. Then, for $L-(b-a) < T < L$, writing the system along the characteristics $(t, t + \lambda x)$, we obtain the relation

$$\int_0^T \Theta(t, t + \lambda x)\, dt = u^1(T + \lambda x) - u^0(\lambda x) \\ - (1 - \lambda)\lambda \left((v^1)'(T + \lambda x) - (v^0)'(\lambda x) \right) \quad \forall x \in [0, (L - T)/\lambda].$$

This shows that, acting by C^1 controls requires C^2 initial and final data.

3.1. Controllability by two controls. The starting point of the proof of Theorem 3.1 is to control system (3.1) with two internal controls. We are going to use the results of [LR03] on the controllability of $n \times n$ quasi-linear systems by n boundary controls and an extension method to obtain the following result. Observe the different levels of regularity between the state and the controls.

PROPOSITION 3.2. *Let us consider the system*

$$\begin{cases} y_t + \Lambda(y)y_x + f(y) = e_1\Theta_1 + e_2\Theta_2, & (t,x) \in [0,T] \times [0,L], \\ y(t,L) = y(t,0), & t \in [0,T], \\ y(0,x) = y^0(x), & x \in [0,L]. \end{cases} \tag{3.8}$$

Assume that (3.3) *holds and let* (3.6) *be satisfied for* $\delta/2$ *(in place of* δ*). Then, for every* $\varepsilon > 0$*, there exists* $\mu > 0$ *such that, for every* $y^0, y^1 \in C^k_L([0,L])^2$ $(k \geq 2)$ *that satisfy*

$$\|y^0\|_{C^k} \leq \mu, \quad \|y^1\|_{C^k} \leq \mu,$$

there exist controls $\Theta_1, \Theta_2 \in C^{k-1}_L([0,T] \times [0,L])$ *that satisfy* (3.5a) *and*

$$\|\Theta_1\|_{C^{k-1}} + \|\Theta_2\|_{C^{k-1}} \leq \varepsilon,$$

and such that the corresponding solution $y \in C^k_L([0,T] \times [0,L])^n$ *to* (3.8) *satisfies*

$$y(T,x) = y^1(x) \quad \forall x \in [0,L], \tag{3.9a}$$

$$\|y\|_{C^k} \leq \varepsilon. \tag{3.9b}$$

Proof. According to (3.2) we can always assume that $\lambda_1(y) < 0 < \lambda_2(y)$ for every $y \in \mathbb{R}^2$, the other two cases being similar. Let us then consider the following boundary control problem on the domain $[0,T] \times [b-\delta, a+\delta+L]$:

$$\begin{cases} y^*_t + \Lambda(y^*)y^*_x + f(y^*) = 0, & (t,x) \in [0,T] \times [b-\delta, a+\delta+L], \\ y^*_1(t, a+\delta+L) = H_1(t), \quad y^*_2(t, b-\delta) = H_2(t), & t \in [0,T], \\ y^*(0,x) = \overline{y^0}(x), & x \in [b-\delta, a+\delta+L], \\ y^*(T,x) = \overline{y^1}(x), & x \in [b-\delta, a+\delta+L], \end{cases} \tag{3.10}$$

where $H_1, H_2 \in C^k([0,T])$ are boundary controls, and where we have extended by periodicity y^0 and y^1 to

$$\overline{y^0}(x) \overset{\text{def}}{=} \begin{cases} y^0(x) & \text{if } x \in [b-\delta, L], \\ y^0(x-L) & \text{if } x \in [L, a+\delta+L], \end{cases}$$

$$\overline{y^1}(x) \overset{\text{def}}{=} \begin{cases} y^1(x) & \text{if } x \in [b-\delta, L], \\ y^1(x-L) & \text{if } x \in [L, a+\delta+L]. \end{cases}$$

Note that since $y_0, y_1 \in C^k_L([0,L])^n$, we have $\overline{y^0}, \overline{y^1} \in C^k([b-\delta, a+\delta+L])^2$. Since, by assumption, T satisfies

$$T > (L - (b-a-2\delta)) \max\left\{\frac{1}{|\lambda_1(0)|}, \frac{1}{|\lambda_2(0)|}\right\},$$

by [LR03, Theorem 1.2], for every $\mu > 0$ small enough, for every y^0, y^1 such that

$$\|y^0\|_{C^k} \le \mu, \quad \|y^1\|_{C^k} \le \mu,$$

there exists $y^* \in C^k([0,T] \times [b-\delta, a+\delta+L])^2$ that satisfies (3.10) with $\|y^*\|_{C^k}$ small. Let y^{**} be any $C^k([0,T] \times [0,L])^2$ function such that $\|y^{**}\|_{C^k} \le \|y^*\|_{C^k}$ and

$$y^{**}(t,x) = \begin{cases} y^*(t,x+L) & \text{if } x \in [0, a+\delta], \\ y^*(t,x) & \text{if } x \in [b-\delta, L]. \end{cases}$$

On the other hand, let us introduce $\overline{y} \in C^k([0,T] \times [0,L])^2$ defined by

$$\overline{y}(t,x) \stackrel{\text{def}}{=} \eta_1(t)u(t,x) + \eta_2(t)v(t,x),$$

where $\eta_1, \eta_2 \in C^\infty([0,T])$ are time cut-off functions with $0 \le \eta_i \le 1$ and

$$\eta_1(0) = 1, \quad \eta_1(T) = 0, \quad \eta_1^{(i)}(0) = \eta_1^{(i)}(T) = 0 \quad \forall i \in [\![1, k+1]\!], \tag{3.11a}$$

$$\eta_2(0) = 0, \quad \eta_2(T) = 1, \quad \eta_2^{(i)}(0) = \eta_2^{(i)}(T) = 0 \quad \forall i \in [\![1, k+1]\!], \tag{3.11b}$$

and $u, v \in C^k([0,T] \times [0,L])^2$ are the solutions to the forward and backward free evolving systems

$$\begin{cases} u_t + \Lambda(u)u_x + f(u) = 0, \quad (t,x) \in [0,T] \times [0,L], \\ u(t,L) = u(t,0), \quad t \in [0,T], \\ u(0,x) = y^0(x), \quad x \in [0,L], \end{cases}$$

and

$$\begin{cases} v_t + \Lambda(v)v_x + f(v) = 0, \quad (t,x) \in [0,T] \times [0,L], \\ v(t,L) = v(t,0), \quad t \in [0,T], \\ v(T,x) = y^1(x), \quad x \in [0,L]. \end{cases}$$

Let now $\xi \in C^\infty([0,L])$ be a space cut-off function with $0 \le \xi \le 1$ and

$$\xi(x) = \begin{cases} 1 & \text{if } x \in [0,a] \cup [b,L], \\ 0 & \text{if } x \in [a+\delta, b-\delta]. \end{cases}$$

Let y and Θ be defined by

$$y(t,x) \stackrel{\text{def}}{=} \xi(x)y^{**}(t,x) + (1-\xi(x))\overline{y}(t,x)$$

and

$$\Theta \stackrel{\text{def}}{=} y_t + \Lambda(y)y_x + f(y). \tag{3.12}$$

By construction, $y \in C^k([0,T] \times [0,L])^2$ and $\Theta \in C^{k-1}([0,T] \times [0,L])^2$. Still by construction, (y, Θ) solves (3.8), y satisfies (3.9a), and Θ satisfies

$$\text{supp}\Theta \subset [0,T] \times [a,b], \tag{3.13a}$$

$$\partial_t^i \Theta(0,\cdot) = \partial_t^i \Theta(T,\cdot) = 0 \quad \forall i \in [\![0, k+1]\!]. \tag{3.13b}$$

The smallnesses of y and Θ follow from the smallnesses of y^* and $\overline{y}$.

To obtain (3.5a) we let system (3.8) evolves freely (without control), forward on the domain $[0,\delta]\times[0,L]$ and backward on the domain $[T-\delta,T]\times[0,L]$, and denote by $y^\delta(x)$ (resp., $y^{T-\delta}(x)$) its value at time $t=\delta$ (resp., $t=T-\delta$). By the previous step, replacing $[a,b]$ by $[a+\delta,b-\delta]$, there exists a control Θ with (3.5a) and

$$\partial_t^i\Theta(\delta,\cdot)=\partial_t^i\Theta(T-\delta,\cdot)=0 \quad \forall i\in[\![0,k+1]\!]$$

that steers the solution to system (3.8), posed on the time reduced domain $[\delta,T-\delta]\times[0,L]$, from y^δ to $y^{T-\delta}$. Thus, we can extend Θ by zero outside $(\delta,T-\delta)\times(a+\delta,b-\delta)$. Note that the smallnesses are preserved. ∎

3.2. Algebraic solvability. In this section we recall the notion of algebraic solvability and the fixed point theorem of Gromov. We refer to [Gro86, section 2.3] for more details.

In what follows Q is a smooth bounded open subset of $\mathbb{R}^2$ and $\mathscr{D}: C^r(\overline{Q})^p \longrightarrow C^0(\overline{Q})^q$ ($p,q\in\mathbb{N}^*$) is a nonlinear C^∞-differential operator of order $r\in\mathbb{N}^*$. We recall that this means that there exists a C^∞-function $F:\mathbb{R}^{n_{r,p}}\longrightarrow\mathbb{R}^q$, where $n_{r,p}=2+p\,\mathrm{card}\,\left\{(\alpha_1,\alpha_2)\in\mathbb{N}^2 \,\middle|\, \alpha_1+\alpha_2\le r\right\}$, such that $\mathscr{D}$ is written

$$\mathscr{D}(z)=F(J^rz)\quad\forall z\in C^r(\overline{Q})^p,$$

where J^rz denotes the r-jet of z, that is, the function defined for every $(t,x)\in\overline{Q}$ by

$$J^rz(t,x)\overset{\text{def}}{=}\left((t,x),z(t,x),\dots,\frac{\partial^{|\alpha|}z}{\partial t^{\alpha_1}\partial x^{\alpha_2}}(t,x),\dots,\frac{\partial^r z}{\partial t^{\alpha_1}\partial x^{\alpha_2}}(t,x)\right)\in\mathbb{R}^{n_{r,p}}.$$

Clearly, the map $\mathscr{D}$ is of class C^∞ and we denote by $\mathscr{L}_z: C^r(\overline{Q})^p\longrightarrow C^0(\overline{Q})^q$ its total differential at $z\in C^r(\overline{Q})^p$.

Definition 3.3. *We say that $\mathcal{A}$ is a differential relation of order d ($d\in\mathbb{N}$) if there exists $\mathcal{R}\subset\mathbb{R}^{n_{d,p}}$ such that*

$$\mathcal{A}\overset{\text{def}}{=}\left\{z\in C^d(\overline{Q})^p\,\middle|\,J^dz(t,x)\in\mathcal{R}\quad\forall(t,x)\in\overline{Q}\right\}.$$

It is said to be open if it is an open subset of $C^d(\overline{Q})^p$.

Definition 3.4. *Let $\mathcal{A}\subset C^d(\overline{Q})^p$ be a differential relation of order d. We say that the operator $\mathscr{D}$ admits an infinitesimal inversion of order $s\in\mathbb{N}$ over $\mathcal{A}$ if there exists a family of linear differential operators of order s, $\mathscr{M}_z: C^s(\overline{Q})^q\longrightarrow C^0(\overline{Q})^p$, $z\in\mathcal{A}$, such that the following hold:*

1. *For every $g\in C^s(\overline{Q})^q$ being fixed, $z\mapsto\mathscr{M}_z(g)$ is a differential operator of order d (possibly nonlinear) and it is a C^∞-differential operator in (z,g).*
2. *Algebraic solvability: for every $z\in\mathcal{A}^{d+s}\overset{\text{def}}{=}\mathcal{A}\cap C^{d+s}(\overline{Q})^p$, we have*

$$\mathscr{L}_z\circ\mathscr{M}_z=\mathrm{Id}_{C^{r+s}(\overline{Q})^q},$$

The proof of Theorem 3.1 is based on the following result [Gro86, section 2.3.2, Main Theorem].

Theorem 3.5. *Let $\mathcal{A}\subset C^d(\overline{Q})^p$ be a nonempty open differential relation of order d. Assume that $\mathscr{D}$ admits an infinitesimal inversion of order s over $\mathcal{A}$. Let*

$$\sigma_0>\max(d,2r+s), \tag{3.14}$$

$$\nu\in(0,+\infty). \tag{3.15}$$

Then, there exist a family of sets $\mathcal{B}_z \subset C^{\sigma_0+s}(\overline{Q})^q$ *and a family of operators* $\mathscr{D}_z^{-1} : \mathcal{B}_z \longrightarrow \mathcal{A}$*, where* $z \in \mathcal{A}^{\sigma_0+r+s}$*, such that the following properties hold:*

1. *Neighborhood property: for every* $z \in \mathcal{A}^{\sigma_0+r+s}$*, we have* $0 \in \mathcal{B}_z$ *and the set* $\mathcal{B}$ *defined by*
$$\mathcal{B} = \bigcup_{z \in \mathcal{A}^{\sigma_0+r+s}} \{z\} \times \mathcal{B}_z$$
is an open subset of $C^{\sigma_0+r+s}(\overline{Q})^p \times C^{\sigma_0+s}(\overline{Q})^q$.
2. *Inversion property:*
$$\mathscr{D}\left(\mathscr{D}_z^{-1}(g)\right) = \mathscr{D}(z) + g \quad \forall\, (z,g) \in \mathcal{B}. \tag{3.16}$$
3. *Normalization property:*
$$\mathscr{D}_z^{-1}(0) = z \quad \forall z \in \mathcal{A}^{\sigma_0+r+s}.$$
4. *Locality: for every* $(t,x) \in \overline{Q}$ *and for every* (z_1,g_1), $(z_2,g_2) \in \mathcal{B}$*, if we have*
$$(z_1,g_1)\,(\tilde{t},\tilde{x}) = (z_2,g_2)\,(\tilde{t},\tilde{x}) \quad \forall (\tilde{t},\tilde{x}) \in \mathrm{B}((t,x),\nu) \cap \overline{Q},$$
then,
$$\mathscr{D}_{z_1}^{-1}(g_1)(t,x) = \mathscr{D}_{z_2}^{-1}(g_2)(t,x).$$

3.3. Proof of Theorem 3.1. Throughout this section, $\delta > 0$ is a fixed real number such that (3.6) holds. Let us denote
$$Q_\delta \stackrel{\text{def}}{=} (\delta, T-\delta) \times (a+\delta, b-\delta),$$
and let Q be an open subset of $\mathbb{R} \times \mathbb{R}$ of class C^∞ such that (see Figure 4)
$$\overline{Q_\delta} \subset Q, \quad \overline{Q} \subset (0,T) \times (a,b).$$

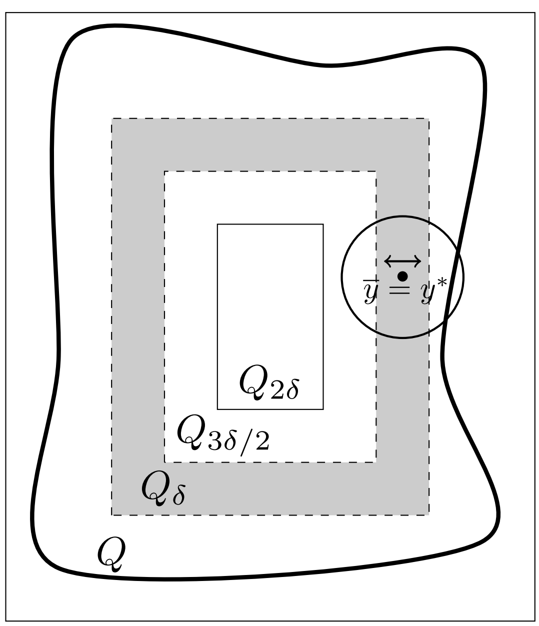

FIG. 4. *Matching* $\overline{y}$ *to* y^*.

Let us introduce the operator $\mathscr{D}$ defined by

$$\begin{array}{rccl} \mathscr{D}: & C^1(\overline{Q})^3 & \longrightarrow & C^0(\overline{Q})^2, \\ & (y,\Theta) & \longmapsto & y_t + \Lambda(y)y_x + f(y) - e_1\Theta. \end{array}$$

This is a nonlinear C^∞-differential operator of order 1. Note that (y,Θ) solves (3.1) if and only if

$$\mathscr{D}(y,\Theta) = 0.$$

Let $\mathscr{L}_{(\tilde{y},\tilde{\Theta})}$ be the differential of the operator $\mathscr{D}$ at $(\tilde{y},\tilde{\Theta}) \in C^1(\overline{Q})^3$:

$$\begin{array}{rccl} \mathscr{L}_{(\tilde{y},\tilde{\Theta})}: & C^1(\overline{Q})^3 & \longrightarrow & C^0(\overline{Q})^2, \\ & (y,\Theta) & \longmapsto & y_t + \Lambda(\tilde{y})y_x + (\Lambda'(\tilde{y})y)\,\tilde{y}_x + f'(\tilde{y})y - e_1\Theta. \end{array}$$

This is a nonlinear C^∞-differential operator of order 1 in $(\tilde{y},\tilde{\Theta})$.

Let us denote

$$y = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \tag{3.17}$$

$$(\Lambda'(\tilde{y})y)\,\tilde{y}_x + f'(\tilde{y})y = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.18}$$

We have

$$a_{11} = \frac{\partial \lambda_1}{\partial u}(\tilde{y})\frac{\partial \tilde{u}}{\partial x} + \frac{\partial f_1}{\partial u}(\tilde{y}), \quad a_{21} = \frac{\partial \lambda_2}{\partial u}(\tilde{y})\frac{\partial \tilde{u}}{\partial x} + \frac{\partial f_2}{\partial u}(\tilde{y}). \tag{3.19}$$

PROPOSITION 3.6. *Let $\mathcal{A}$ be defined by*

$$\mathcal{A} \overset{\text{def}}{=} \left\{ \left(\tilde{y},\tilde{\Theta}\right) \in C^2(\overline{Q})^3 \,\middle|\, a_{21}(t,x) \neq 0 \quad \forall (t,x) \in \overline{Q} \right\}. \tag{3.20}$$

Then, $\mathcal{A}$ is an open differential relation of order 2 *and the operator $\mathscr{D}$ admits an infinitesimal inversion of order* 1 *over $\mathcal{A}$.*

Proof. Clearly, $\mathcal{A}$ is a differential relation, associated with the set $\mathcal{R}$ consisting of the points

$$\left((t,x), \begin{pmatrix} \alpha_{00} \\ \beta_{00} \\ \gamma_{00}, \end{pmatrix}, \begin{pmatrix} \alpha_{10} \\ \beta_{10} \\ \gamma_{10}, \end{pmatrix}, \begin{pmatrix} \alpha_{01} \\ \beta_{01} \\ \gamma_{01}, \end{pmatrix}, \begin{pmatrix} \alpha_{20} \\ \beta_{20} \\ \gamma_{20}, \end{pmatrix}, \begin{pmatrix} \alpha_{11} \\ \beta_{11} \\ \gamma_{11}, \end{pmatrix}, \begin{pmatrix} \alpha_{02} \\ \beta_{02} \\ \gamma_{02}, \end{pmatrix} \right) \in \overline{Q} \times \mathbb{R}^{3\times 6},$$

such that

$$\frac{\partial \lambda_2}{\partial u} \begin{pmatrix} \alpha_{00} \\ \beta_{00} \end{pmatrix} \alpha_{01} + \frac{\partial f_2}{\partial u} \begin{pmatrix} \alpha_{00} \\ \beta_{00} \end{pmatrix} \neq 0.$$

We can check that $\mathcal{A}$ is open in $C^2(\overline{Q})^3$.

Let now $(\tilde{y}, \tilde{\Theta}) \in \mathcal{A}$. Let $(g_1, g_2) \in C^1(\overline{Q})^2$. We have to solve the equation $\mathscr{L}_{(\tilde{y},\tilde{\Theta})}(y, \Theta) = (g_1, g_2)$ in such a way that (y, Θ) is a linear combination of derivatives of g_1 and g_2. The equation $\mathscr{L}_{(\tilde{y},\tilde{\Theta})}(y, \Theta) = (g_1, g_2)$ is rewritten as

$$\begin{cases} u_t + \lambda_1(\tilde{y})u_x + a_{11}u + a_{12}v - \Theta &= g_1 \quad \text{in } \overline{Q}, \\ v_t + \lambda_2(\tilde{y})v_x + a_{21}u + a_{22}v &= g_2 \quad \text{in } \overline{Q}. \end{cases}$$

By definition of $\mathcal{A}$, we have

$$a_{21}(t, x) \neq 0 \quad \forall (t, x) \in \overline{Q}. \tag{3.21}$$

In this case, the algebraic solvability is not very difficult: we first put

$$v = 0, \tag{3.22}$$

so that the second equation simply becomes $a_{21}u = g_2$. As a result, using (3.21) we can take

$$u = \frac{1}{a_{21}} g_2. \tag{3.23}$$

Finally, it remains to set

$$\Theta = -g_1 + \frac{1}{a_{21}} \left((g_2)_t + \lambda_1(\tilde{y})(g_2)_x\right) + \left(\left(\frac{1}{a_{21}}\right)_t + \lambda_1(\tilde{y}) \left(\frac{1}{a_{21}}\right)_x + \frac{a_{11}}{a_{21}}\right) g_2. \tag{3.24}$$

(Note that (3.24), together with (3.19), shows why $C^2(\overline{Q})^3$ cannot be replaced by $C^1(\overline{Q})^3$ in Proposition 3.6.)

Then, the following family of differential operators satisfies all the required properties ($s = 1$) to be an infinitesimal inversion of $\mathscr{D}$ over $\mathcal{A}$:

$$\begin{aligned} \mathscr{M}_{(\tilde{y},\tilde{\Theta})} : \quad C^1(\overline{Q})^2 &\longrightarrow C^0(\overline{Q})^3, \\ (g_1, g_2) &\longmapsto ((u, v), \Theta), \end{aligned}$$

where u, v and Θ are respectively defined by (3.23), (3.22), and (3.24). □

Proof of Theorem 3.1. Let $\varepsilon > 0$ be fixed. Let $\mathcal{A}$ be the open differential relation of order 2 defined by (3.20) and set

$$\mathcal{A}_\varepsilon \stackrel{\text{def}}{=} \mathcal{A} \cap \left\{ (\tilde{y}, \tilde{\Theta}) \in C^1(\overline{Q})^3 \;\middle|\; \|\tilde{y}\|_{C^1} < \varepsilon,\, \|\tilde{\Theta}\|_{C^1} < \varepsilon \right\}. \tag{3.25}$$

Note that $\mathcal{A}_\varepsilon$ is an open differential relation of order 2 which is nonempty since $0 \in \mathcal{A}_\varepsilon$ by assumption (3.4). We choose

$$\nu \stackrel{\text{def}}{=} \frac{\delta}{2}, \tag{3.26}$$

$$d = 2,\ r = 1,\ s = 1,\ \sigma_0 = 4. \tag{3.27}$$

Note that from (3.26) one gets (3.15), and from (3.27) one gets (3.14).

Thanks to Proposition 3.6, $\mathscr{D}$ admits an infinitesimal inversion of order 1 over $\mathcal{A}_\varepsilon$. Thus, we can apply Theorem 3.5, which provides a family of sets $\mathcal{B}_{(\tilde{y},\tilde{\Theta})} \subset C^5(\overline{Q})^2$

and a family of operators $\mathscr{D}^{-1}_{(\tilde{y},\tilde{\Theta})} : \mathcal{B}_{(\tilde{y},\tilde{\Theta})} \longrightarrow \mathcal{A}_\varepsilon$, where $(\tilde{y},\tilde{\Theta}) \in \mathcal{A}^6_\varepsilon \stackrel{\text{def}}{=} \mathcal{A}_\varepsilon \cap C^6(\overline{Q})^3$, such that all the properties listed in this theorem hold.

Since $0 \in \mathcal{A}_\varepsilon$, $\mathcal{B}$ also contains 0. Since $\mathcal{B}$ is open, there exists $\rho > 0$ such that

(3.28)
$$\left\{\left(\left(\tilde{y},\tilde{\Theta}\right),g\right) \in C^6(\overline{Q})^3 \times C^5(\overline{Q})^2 \;\middle|\; \|\tilde{y}\|_{C^6} \le \rho,\ \|\tilde{\Theta}\|_{C^6} \le \rho,\ \|g\|_{C^5} \le \rho\right\} \subset \mathcal{B}.$$

On the other hand, by Proposition 3.2, there exists $\mu > 0$ such that, for every $y^0, y^1 \in C^6_L([0,L])^2$ that satisfy
$$\|y^0\|_{C^6} \le \mu, \quad \|y^1\|_{C^6} \le \mu,$$
there exist $\Theta^*_1, \Theta^*_2 \in C^5([0,T]\times[0,L])$ and $y^* \in C^6([0,T]\times[0,L])^2$ such that
$$\mathscr{D}(y^*,0) = -e_1\Theta^*_1 - e_2\Theta^*_2,$$
$$y^*(T,x) = y^1(x) \quad \forall x \in [0,L],$$
$$\text{supp}\Theta^*_1 \subset Q_{2\delta}, \quad \text{supp}\Theta^*_2 \subset Q_{2\delta},$$
and
$$\|y^*\|_{C^6} < \varepsilon', \quad \|\Theta^*_1\|_{C^5} + \|\Theta^*_2\|_{C^5} < \varepsilon',$$
where $\varepsilon' = \min(\varepsilon,\rho)$. In particular,
$$(y^*, 0, -e_1\Theta^*_1 - e_2\Theta^*_2) \in \mathcal{B},$$
and we can set
$$(\overline{y},\overline{\Theta}) \stackrel{\text{def}}{=} \mathscr{D}^{-1}_{(y^*,0)}\left(-e_1\Theta^*_1 - e_2\Theta^*_2\right).$$
By definition $(\overline{y},\overline{\Theta}) \in C^1(\overline{Q})^3$ and satisfies
$$\mathscr{D}(\overline{y},\overline{\Theta}) = \mathscr{D}(y^*,0) - e_1\Theta^*_1 - e_2\Theta^*_2 = 0.$$

Let us now prove that we can match $\overline{y}$ to y^* and $\overline{\Theta} = 0$ in the open neighborhood $Q_\delta \backslash \overline{Q}_{3\delta/2}$ (see Figure 4). Let then $(t,x) \in Q_\delta\backslash\overline{Q}_{3\delta/2}$ be fixed and let us show that

$$\overline{y}(t,x) = y^*(t,x), \quad \overline{\Theta}(t,x) = 0. \tag{3.29}$$

Since Θ^*_1 and Θ^*_2 are supported in $\overline{Q}_{2\delta}$, we have
$$(y^*, 0, -e_1\Theta^*_1 - e_2\Theta^*_2)(\tilde{t},\tilde{x}) = (y^*,0,0)(\tilde{t},\tilde{x}) \quad \forall(\tilde{t},\tilde{x}) \in \mathrm{B}((t,x),\delta/2),$$
and $(y^*, 0, -e_1\Theta^*_1 - e_2\Theta^*_2), (y^*,0,0) \in \mathcal{B}$. Thus, by (3.26), the locality and normalization properties, we obtain (3.29). To conclude the proof, it remains to set
$$y(t,x) \stackrel{\text{def}}{=} \begin{cases} \overline{y}(t,x) & \text{if } (t,x) \in Q_\delta, \\ y^*(t,x) & \text{if } (t,x) \in [0,T]\times[0,L]\backslash Q_\delta, \end{cases}$$
and
$$\Theta(t,x) \stackrel{\text{def}}{=} \begin{cases} \overline{\Theta}(t,x) & \text{if } (t,x) \in Q_\delta, \\ 0 & \text{if } (t,x) \in [0,T]\times[0,L]\backslash Q_\delta. \end{cases}$$
Then, (y,Θ) solves (3.1). Since $y = y^*$ near the boundary of $[0,T]\times[0,L]$ it satisfies the same periodic boundary conditions $y(\cdot,L) = y(\cdot,0)$, the same initial condition $y(0,\cdot) = y^0$, and the same final condition $y(T,\cdot) = y^1$. Finally, note that the smallnesses of y and Θ follow from the smallness of y^* and the definition (3.25) of $\mathcal{A}_\varepsilon$. ∎

Appendix A. Proof of Lemma 2.6. Let us compute $\tau_k(x,b)-\tau_k(x,a)$ for every $k \in [\![k_{\min}(x,b), k_{\max}(x,a)]\!]$ and for every $x \in [0,L)$,

$$\tau_k(x,b)-\tau_k(x,a) = \begin{cases} b-x+kL & \text{if } a-x+kL \in (-\infty,0) \quad \text{and } b-x+kL \in [0,T], \\ T & \text{if } a-x+kL \in (-\infty,0) \quad \text{and } b-x+kL \in (T,+\infty), \\ b-a & \text{if } a-x+kL \in [0,T] \quad \text{and } b-x+kL \in [0,T], \\ T-a+x-kL & \text{if } a-x+kL \in [0,T] \quad \text{and } b-x+kL \in (T,+\infty). \end{cases}$$

Thus, we see that the main problem that we have to handle is the fact that the length $\tau_0(x,b)-\tau_0(x,a)$ (resp., $\tau_{\lceil \frac{T-a}{L} \rceil}(x,b) - \tau_{\lceil \frac{T-a}{L} \rceil}(x,a)$) goes to zero as x goes to b^- (resp., $p(a)^+$). We are going to approximate uniformly the Gramian associated with (2.4)–(2.5) by modifying it a little bit near the points b and $p(a)$ in order to avoid the aforementioned problems. This has to be done in such a way that the invertibility is preserved. Let

$$\ell_0 = \min\left(T-a, b-a, L-p(a), b-L+\left\lceil \frac{T-a}{L} \right\rceil L\right)/2$$

(note that ℓ_0 is positive thanks to $(\mathcal{H}_1)$). Then, for every $x \in [0,L)$ and $0 \leq \ell < \ell_0$ we introduce the approximated Gramian

$$Q(x,\ell) = Q_0(x,\ell) + \sum_{k=1}^{\lceil \frac{T-a}{L} \rceil - 1} Q_k(x,\ell) + Q_{\lceil \frac{T-a}{L} \rceil}(x,\ell),$$

where, for every $k \in [\![1, \lceil \frac{T-a}{L} \rceil - 1]\!]$,

$$Q_k(x,\ell) = \int_{\tau_k(x,a)+\ell}^{\tau_k(x,b)-\ell} G(s,x)\, ds,$$

with $G(s,x) = R^x(T,s)\, B^x(s) B^x(s)^* R^x(T,s)^*$, and

$$Q_0(x,\ell) = \begin{cases} \displaystyle\int_{\tau_0(x,a)+\ell}^{\tau_0(x,b)-\ell} G(s,x)\, ds & \text{if } x \in [0, b-2\ell), \\ 0 & \text{if } x \in [b-2\ell, L), \end{cases}$$

and where

$$Q_{\lceil \frac{T-a}{L} \rceil}(x,\ell) = \begin{cases} 0 & \text{if } x \in [0, p(a)+2\ell], \\ \displaystyle\int_{\tau_{\lceil \frac{T-a}{L} \rceil}(x,a)+\ell}^{\tau_{\lceil \frac{T-a}{L} \rceil}(x,b)-\ell} G(s,x)\, ds & \text{if } x \in (p(a)+2\ell, L). \end{cases}$$

Step 1. First, let us prove that there exists $r > 0$ such that, for every $x \in [0,L)$,

$$\mathrm{B}\left(Q(x,0), r\right) \subset \mathrm{GL}_n(\mathbb{R}), \tag{A.1}$$

where $\mathrm{GL}_n(\mathbb{R})$ denotes again the set of invertible matrices of size n and $\mathrm{B}(M,\rho)$ the open ball of center $M \in \mathbb{R}^{n\times n}$ and radius $\rho > 0$. Since $\lim_{\substack{x \to L \\ x < L}} Q(x,0) = Q(0,0)$, we can extend $Q(\cdot,0)$ by continuity to $[0,L]$ (still denoted by $Q(\cdot,0)$) with $Q(L,0)=Q(0,0)$. By assumption $(\mathcal{H}_2)$, for every $x\in[0,L]$, $Q(x,0)\in \mathrm{GL}_n(\mathbb{R})$. Since $\mathrm{GL}_n(\mathbb{R})$ is open, there exists $r(x)>0$ such that

$$\mathrm{B}\left(Q(x,0),r(x)\right)\subset \mathrm{GL}_n(\mathbb{R}). \tag{A.2}$$

On the other hand,

$$Q([0,L],0)\subset \bigcup_{x\in[0,L]} \mathrm{B}\left(Q(x,0),\frac{r(x)}{2}\right),$$

and $Q([0,L],0)$ is compact (by continuity of $Q(\cdot,0)$ on all $[0,L]$), so that there exists $x_1,\dots,x_q\in[0,L]$ such that

$$Q([0,L],0)\subset \bigcup_{x\in\{x_1,\dots,x_q\}} \mathrm{B}\left(Q(x,0),\frac{r(x)}{2}\right).$$

We define

$$r=\min\left\{\frac{r(x_1)}{2},\dots,\frac{r(x_q)}{2}\right\}.$$

Thus, for every $x\in[0,L]$, for every $M\in\mathrm{B}\left(Q(x,0),r\right)$, there exists $x_i\in\{x_1,\dots,x_q\}$ such that

$$\|M-Q(x_i,0)\|\leq\|M-Q(x,0)\|+\|Q(x,0)-Q(x_i,0)\|<r+\frac{r(x_i)}{2}\leq r(x_i),$$

that is, $M\in\mathrm{B}(Q(x_i,0),r(x_i))$ and shows that M is invertible by (A.2).

Step 2. By construction we have

$$\sup_{x\in[0,L)}\|Q(x,\ell)-Q(x,0)\|\xrightarrow[\ell\to 0]{}0.$$

Thus, there exists $\delta>0$ small enough such that, for every $x\in[0,L)$, we have

$$Q(x,2\delta)\in\mathrm{B}\left(Q(x,0),r\right),$$

which shows that $Q(x,2\delta)$ is invertible by (A.1).

Step 3. Let us now define the cut-off function η (see Figure 3). Let us introduce

$$\mathcal{T}_0(\delta)\overset{\text{def}}{=}\{(t,x)\in(0,T)\times(0,L)\quad|\quad x\in(0,b-2\delta),\quad t\in(\tau_0(x,a)+\delta,\tau_0(x,b)-\delta)\},$$

$$\mathcal{T}_{\lceil\frac{T-a}{L}\rceil}(\delta)\overset{\text{def}}{=}\Big\{(t,x)\in(0,T)\times(0,L),\quad\Big|\quad x\in(p(a)+2\delta,L),\quad t\in\left(\tau_{\lceil\frac{T-a}{L}\rceil}(x,a)+\delta,\tau_{\lceil\frac{T-a}{L}\rceil}(x,b)-\delta\right)\Big\},$$

and, for $k\in[\![1,\lceil\frac{T-a}{L}\rceil-1]\!]$,

$$\mathcal{T}_k(\delta)\overset{\text{def}}{=}\{(t,x)\in(0,T)\times(0,L)\quad|\quad t\in(\tau_k(x,a)+\delta,\tau_k(x,b)-\delta)\}.$$

Let $\xi \in C^1([0,T]\times[0,L])$ be a cut-off function with $0 \leq \xi \leq 1$ and such that (see Figure 2 to help)

$$\xi \equiv 1 \text{ in } \bigcup_{k=0}^{\lceil \frac{T-a}{L} \rceil} \mathcal{T}_k(2\delta), \tag{A.3a}$$

$$\xi \equiv 1 \text{ in } \bigcup_{k=0}^{\lceil \frac{T-a}{L} \rceil -1} \left\{(t,0) \,|\, t \in (\tau_k(0,a)+2\delta, \tau_k(0,b)-2\delta)\right\}, \tag{A.3b}$$

$$\xi \equiv 1 \text{ in } \bigcup_{k=1}^{\lceil \frac{T-a}{L} \rceil} \left\{(t,L) \,|\, t \in (\tau_{k-1}(0,a)+2\delta, \tau_{k-1}(0,b)-2\delta)\right\}, \tag{A.3c}$$

$$\xi \equiv 0 \text{ in } [0,T]\times[0,L] \setminus \left(\bigcup_{k=0}^{\lceil \frac{T-a}{L} \rceil} \overline{\mathcal{T}_k(\delta)} \right). \tag{A.3d}$$

For every $x \in [0,L]$, let us define

$$Q_x = \int_0^T G(s,x)\xi(s,x)\,ds.$$

Note that $Q_L = Q_0$. Let us show that Q_x is invertible for every $x \in [0,L]$. First, note that the controllability Gramian Q_x is a nonnegative symmetric matrix. Thus, it is invertible if and only if it is positive definite. Let $v \in \mathbb{R}^n$. Using $\xi \geq 0$ and (A.3a)–(A.3d), we have

$$\begin{aligned} v \cdot Q_x v &= \int_0^T \|B(s)^* R(T,s)^* v\|^2 \xi(s,x)\,ds \\ &= \sum_{k=0}^{\lceil \frac{T-a}{L} \rceil} \int_{\left\{t\in[0,T] \,\middle|\, (t,x)\in\overline{\mathcal{T}_k(\delta)}\right\}} \|B(s)^* R(T,s)^* v\|^2 \xi(s,x)\,ds \\ &\geq \sum_{k=0}^{\lceil \frac{T-a}{L} \rceil} \int_{\{t\in[0,T] \,|\, (t,x)\in\mathcal{T}_k(2\delta)\}} \|B(s)^* R(T,s)^* v\|^2 \xi(s,x)\,ds = v \cdot Q(x,2\delta)v. \end{aligned}$$

As a result, the positive definiteness of Q_x follows from the one of $Q(x,2\delta)$.

To conclude, it remains to set, for every $(t,x) \in [0,T]\times[0,L]$,

$$\eta(t,x) \stackrel{\text{def}}{=} \begin{cases} \xi(t, \overline{X}^{-1}(t,x)) & \text{if } (t,x) \in \mathcal{T}', \\ \xi(t,0) & \text{if } x = \overline{X}(t,0), \\ 0 & \text{if } (t,x) \in \partial\left([0,T]\times[0,L]\right). \end{cases}$$

Thanks to (A.3d), we have $\eta \in C^1([0,T]\times[0,L])$ and (2.11). Moreover,

$$\eta\left(t, \overline{X}(t,x)\right) = \xi(t,x), \quad \forall (t,x) \in [0,T]\times[0,L].$$

REFERENCES

[AB03] F. Alabau-Boussouira, *A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems*, SIAM J. Control Optim., 42 (2003), pp. 871–906.

[AB13] F. Alabau-Boussouira, *A hierarchic multi-level energy method for the control of bidiagonal and mixed n-coupled cascade systems of PDE's by a reduced number of controls*, Adv. Differential Equations, 18 (2013), pp. 1005–1072.

[AB14] F. Alabau-Boussouira, *Insensitizing exact controls for the scalar wave equation and exact controllability of 2-coupled cascade systems of PDE's by a single control*, Math. Control Signals Systems, 26 (2014), pp. 1–46.

[ABL12] F. Alabau-Boussouira and M. Léautaud, *Indirect stabilization of locally coupled wave-type systems*, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 548–582.

[AKBGBdT11] F. A. Khodja, A. Benabdallah, M. González-Burgos, and L.de Teresa, *Recent results on the controllability of linear coupled parabolic problems: A survey*, Math. Control Relat. Fields, 1 (2011), pp. 267–306.

[Ala01] F. Alabau, *Observabilité frontière indirecte de systèmes faiblement couplés*, C. R. Acad. Sci. Paris Sér. I Math., 333 (2001), pp. 645–650.

[BFH15] P. Baldi, G. Floridia, and E. Haus, *Exact controllability for quasi-linear perturbations of KdV*, Anal. PDE, to appear.

[CBdN08] J.-M. Coron, G. Bastin, and B. d'Andréa Novel, *Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems*, SIAM J. Control Optim., 47 (2008), pp. 1460–1498.

[CG09a] J.-M. Coron and S. Guerrero, *Local null controllability of the two-dimensional Navier-Stokes system in the torus with a control force having a vanishing component*, J. Math. Pures Appl. (9), 92 (2009), pp. 528–545.

[CG09b] J.-M. Coron and S. Guerrero, *Null controllability of the N-dimensional Stokes system with N-1 scalar controls*, J. Differential Equations, 246 (2009), pp. 2908–2921.

[CGR10] J.-M. Coron, S. Guerrero, and L. Rosier, *Null controllability of a parabolic system with a cubic coupling term*, SIAM J. Control Optim., 48 (2010), pp. 5629–5653.

[Cir69] M. Cirinà, *Boundary controllability of nonlinear hyperbolic systems*, SIAM J. Control, 7 (1969), pp. 198–212.

[CL14] J.-M. Coron and P. Lissy, *Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components*, Invent. Math., 198 (2014), pp. 833–880.

[Cor92] J.-M. Coron, *Global asymptotic stabilization for controllable systems without drift*, Math. Control Signals Systems, 5 (1992), pp. 295–312.

[Cor07] J.-M. Coron, *Control and Nonlinearity*, Math. Surveys Monogr. 136, AMS, Providence, RI, 2007.

[Dág06] R. Dáger, *Insensitizing controls for the 1-d wave equation*, SIAM J. Control Optim., 45 (2006), pp. 1758–1768.

[DL16a] M. Duprez and P. Lissy, *Indirect controllability of some linear parabolic systems of m equations with m - 1 controls involving coupling terms of zero or first order*, J. Math. Pures Appl., 106 (2016), pp. 905–934.

[DL16b] M. Duprez and P. Lissy, *Positive and negative results on the internal controllability of parabolic equations coupled by zero and first order terms*, submitted.

[DLRL14] B. Dehman, J. L. Rousseau, and M. Léautaud, *Controllability of two coupled wave equations on a compact manifold*, Arch. Ration. Mech. Anal., 211 (2014), pp. 113–187.

[dT00] L. d. Teresa, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations, 25 (2000), pp. 39–72.

[FCGIP06] E. F. Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel, *Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with N-1 scalar controls*, SIAM J. Control Optim., 45 (2006), pp. 146–173.

[GBPG05] M. G. Burgos and R. P. García, *Controllability of some coupled parabolic systems by one control force*, C. R. Math. Acad. Sci. Paris, 340 (2005), pp. 125–130.

[Gro86] M. Gromov, *Partial Differential Relations*, Ergeb. Math. Grenzgeb. (3), 9, Springer-Verlag, Berlin, 1986.

[Gue07] S. Guerrero, *Null controllability of some systems of two parabolic equations with one control force*, SIAM J. Control Optim., 46 (2007), pp. 379–394.

[Har82] P. Hartman, *Ordinary Differential Equations*, 2nd ed., Birkhäuser, Boston, Mass., 1982.

[KHN63] R. E. Kalman, Y.-C. Ho, and K. S. Narendra, *Controllability of linear dynamical systems*, Contrib. Differential Equations, 1 (1963), pp. 189–213.

[LJ01] T. Li and Y. Jin, *Semi-global C^1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 22 (2001), pp. 325–336.

[LR02] T. Li and B. Rao, *Local exact boundary controllability for a class of quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 23 (2002), pp. 209–218.

[LR03] T. Li and B. Rao, *Exact boundary controllability for quasi-linear hyperbolic systems*, SIAM J. Control Optim., 41 (2003), pp. 1748–1755.

[LRW10] T. Li, B. Rao, and Z. Wang, *Exact boundary controllability and observability for first order quasilinear hyperbolic systems with a kind of nonlocal boundary conditions*, Discrete Contin. Dyn. Syst., 28 (2010), pp. 243–257.

[LY85] T. Li and W. C. Yu, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke Univ. Math. Ser. 5, Duke University, Durham, NC, 1985.

[SM67] L. M. Silverman and H. E. Meadows, *Controllability and observability in time-variable linear systems*, SIAM J. Control, 5 (1967), pp. 64–73.

[Wan06] Z. Wang, *Exact controllability for nonautonomous first order quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 27 (2006), pp. 643–656.

[Zha09] Q. Zhang, *Exact boundary controllability with less controls acting on two ends for quasilinear hyperbolic systems*, Appl. Math. J. Chinese Univ. Ser. A, 24 (2009), pp. 65–74.